MST204.28

The Open University
Mathematics/Science/Technology
An Inter-faculty Second Level Course
MST204 Mathematical Models and Methods

mathematical models and methods

Unit 28 Moments and circular motion

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Unit 28
Moments and circular motion

Prepared by the Course Team

The Open University

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Contents

Introduction

The theme of this unit is the description of turning motion, and of the physical effects which cause it. There is some initial revision of earlier topics, and towards the end of the text the ground is prepared for the study of the two remaining mechanics units.

In *Unit 17* you saw how the particle mechanics of *Units 4*, *7*, *8* and *15* may be applied to systems of particles or to extended objects. Instead of imagining a physical object to be so small that it can reasonably be thought of as a particle, the finite size of the object was acknowledged. However, for the purposes of describing the motion of such an object, we concentrated attention on its centre of mass, since this point moves in the same way as a particle which has the same total mass as the object and which is acted upon by the same total force.

This mathematical model, although useful, does not tell the whole story when it comes to describing the motion of an extended object. For example, the motion of a ball thrown through the air is in part described by saying that, neglecting air resistance, the centre of mass of the ball has a parabolic trajectory (as predicted in *Unit 15*), but this statement takes no account of any spin which the ball may have.

We shall now develop a model which permits the description of such rotational motion for rigid extended objects. As for the case of linear motion in *Unit 4*, we seek in addition to this kinematical description a means of identifying the physical effects which cause objects to rotate. These turning effects are called *moments*.

For example, in turning on a tap you apply with your fingers forces on the bars at the head of the tap. These forces provide moments about the vertical axis of the tap, resulting in a turning motion about this axis. The size of the turning effect caused depends both on how hard you push and on how close to the central axis of the tap your fingers are located. This demonstrates that the particular point of application of a force on an object is significant in terms of what turning effect is produced.

The precise relationship between moments (or *torques*) and rotational motion will be derived in *Unit 29*, and may be thought of as a rotational version of Newton's second law. Here we shall proceed instead by applying Newton's second law directly where necessary.

We shall also consider the conditions which are necessary to ensure that an extended object remains *static*. As for the particle model, the total external force on a static object must be zero, but now an additional rule of a similar nature is required for the moments involved. The first condition ensures that the centre of mass of the object remains fixed, while the second corresponds to an absence of rotation.

Study guide

The first section looks again at the idea of expressing a force in terms of components, though the treatment differs slightly from that in *Unit 15*. The conditions for a particle to be static are also reviewed. The audio-tape (Subsection 1.3) introduces the difference between the effects of forces acting on a particle and on an extended object.

Section 2 introduces a mathematical description of the moment or turning effect of a force, and derives conditions on the forces and moments acting on an object which are sufficient to ensure that the object remains static.

In Section 3 the unit takes a different tack, using plane polar coordinates to describe the position, velocity and acceleration of a particle which moves in a circle.

Thus far, vector notation has been largely avoided. In Section 4 the foregoing ideas are re-expressed in vector terms, as a preliminary to the vector treatment in *Unit 29*. The television programme is associated with Subsection 4.2, and again deals with circular motion. You will find it helpful to have studied at least Sections 1 and 3 before viewing the programme.

Section 5 contains a variety of revision exercises on Sections 2 and 3 of the unit.

1 The components of a force

We aim to cover three topics in this section, two of which involve revision. Following a second look at forces and components, there is a reminder of what it means mathematically for a particle to be *static* or in static *equilibrium*. Finally there is some motivation for the following sections, with a brief discussion (on audio-tape) of the inadequacy of the particle model when the turning motion of an extended object is involved.

Unit 15 Subsections 1.5 and 3.1

To start with, you may recall from *Unit 15* that, with respect to a static Cartesian coordinate system, Newton's second law $m\ddot{\mathbf{r}} = \mathbf{F}$ can be written in the form

Unit 15 Subsection 2.1

$$m(\ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k}) = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}.$$

In order to integrate this equation of motion for a given force $\mathbf{F}$, it is often convenient to equate the corresponding components on both sides of the equation, which gives the three scalar equations

$$m\ddot{x} = F_x, \quad m\ddot{y} = F_y, \quad m\ddot{z} = F_z.$$

In many problems, and especially when the situation is two-dimensional, it is possible to write down the (scalar) component equations of motion without first writing down the vector equation. For example, in the shot-putter's problem which was considered in the television programme for *Unit 15*, it is relatively simple to write down directly from Figure 1 the three component equations of motion

Unit 15 Subsection 4.2

$$m\ddot{x} = 0, \quad m\ddot{y} = -mg, \quad m\ddot{z} = 0.$$

These equations can be integrated individually to give

$$x = (v_0 \cos\theta)t, \quad y = -\tfrac{1}{2}gt^2 + (v_0 \sin\theta)t, \quad z = 0,$$

after applying the initial conditions to find the constants of integration. In order to use this technique for cases where the forces are not directed parallel to the Cartesian axes, it is necessary to find the components of a force of known magnitude and direction.

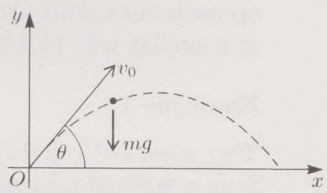


Figure 1

1.1 The resolution of a force

Force is a vector quantity, since it has both magnitude and direction. In *Unit 15* you saw how to describe a force in terms of the Cartesian unit vectors. In this subsection we shall explain what is meant by *resolving* a force or, putting it less mathematically, saying how much of a given force acts in a specified direction or directions. To keep things simple only two-dimensional problems will be considered, although the method developed is applicable also in three dimensions.

Consider a two-dimensional force $\mathbf{F}$ of magnitude $F = |\mathbf{F}|$, which acts in a direction making an anticlockwise angle α with the positive x-axis, as shown in Figure 2. The expression for the force $\mathbf{F}$ in terms of the Cartesian unit vectors is

$$\mathbf{F} = F\cos\alpha\,\mathbf{i} + F\sin\alpha\,\mathbf{j}.$$

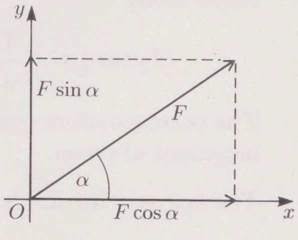


Figure 2

The *components* of the force $\mathbf{F}$ in the directions Ox and Oy are respectively $F_x = F\cos\alpha$ and $F_y = F\sin\alpha$. In other words, the force $\mathbf{F}$ may be considered as the sum of a force with x-component $F\cos\alpha$ in the direction of the x-axis and a force with y-component $F\sin\alpha$ in the direction of the y-axis. This process of finding the components of a force in orthogonal directions is called the *resolution* of the force.

Note that the x-component of the force $\mathbf{F}$ is equal to the magnitude of the force times the cosine of the angle between the direction of the force and the x-axis. This result is restated below in a form which shows that it applies even when the direction of resolution involved is not the x-axis.

The component of a force

The component of a force $\mathbf{F}$ in a direction making an angle α with that of the force is equal to $F\cos\alpha$, where $F = |\mathbf{F}|$ (see Figure 3).

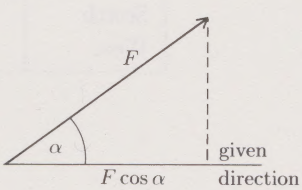


Figure 3

This result can in particular be used to find the y-component of the force $\mathbf{F}$ since, as illustrated by Figure 4,

$$F\cos(\tfrac{1}{2}\pi - \alpha) = F\sin\alpha.$$

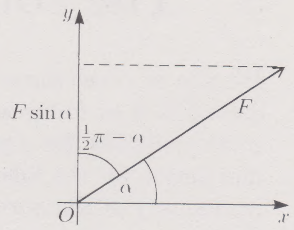


Figure 4

Exercise 1

For the force shown in Figure 2, use the result in the box on the previous page to find the components of the force in the directions of the negative x-axis and of the negative y-axis.

[*Solution on page 38*]

For the force and four directions shown in Figure 5, it has been shown in Exercise 1 and the preceding text that the components of the force are as given in Table 1.

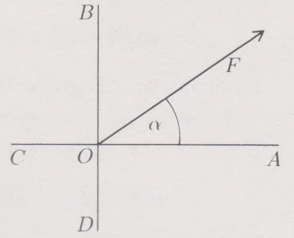


Figure 5

Table 1

Direction	Component of force
OA	$F\cos\alpha$
OB	$F\sin\alpha$
OC	$-F\cos\alpha$
OD	$-F\sin\alpha$

The formulas in Table 1 hold for all values of the angle α, but are usually applied in cases where the angle α is chosen to be acute. Note that the components of the force in the directions OC and OD are the negatives of the components in the respective opposite directions OA and OB. The result in the box prior to Exercise 1 can be used in a similar way to find the component of a force in any direction.

Example 1

Two wires OW_1 and OW_2 radiate horizontally (that is, parallel to the level ground) from a vertical telegraph pole O in the northerly and south-easterly directions, as shown from a bird's-eye view in Figure 6. The wires have tensions T_1 and T_2 respectively. Find the components of the forces acting on the pole in the northerly, easterly, southerly and westerly directions.

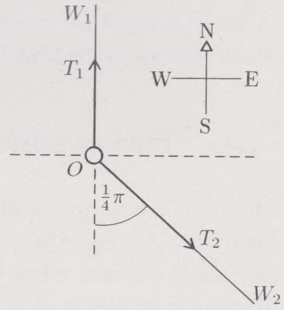


Figure 6

Solution

The tension T_1 in the wire OW_1 produces a force on the pole in a northerly direction. Its component in the northerly direction is therefore just T_1, and its component in the easterly direction is zero. The components in the southerly and westerly directions are the respective negatives of these.

The tension T_2 in the wire OW_2 leads to a force on the pole in the south-easterly direction. The components of this force in the southerly and easterly directions are respectively

$$T_2\cos\tfrac{1}{4}\pi = \frac{T_2}{\sqrt{2}} \quad \text{and} \quad T_2\sin\tfrac{1}{4}\pi = \frac{T_2}{\sqrt{2}}.$$

The corresponding components in the northerly and westerly directions are the negatives of these.

The answers for the force components are summarized in Table 2.

Table 2

Direction	Wire OW_1	Wire OW_2
North	T_1	$-T_2/\sqrt{2}$
East	0	$T_2/\sqrt{2}$
South	$-T_1$	$T_2/\sqrt{2}$
West	0	$-T_2/\sqrt{2}$

□

Exercise 2

A force of magnitude F acts in the direction North-East. Find the components of this force in the directions East and North-West.

Exercise 3

Figure 7 shows a particle P of mass m at rest on a rough plane, which is inclined at an angle θ to the horizontal. You may recall from *Unit 15* Subsection 3.2 that there are three forces acting on the particle, namely

(i) the force of gravity, of magnitude mg acting vertically downwards;

(ii) the normal reaction from the plane, of magnitude N (denoted by $|\mathbf{F}_N|$ in *Unit 15*) in the direction of the upward normal to the plane;

(iii) the force of friction, of magnitude S (denoted by $|\mathbf{F}_f|$ in *Unit 15*) in the direction directly up the plane.

Find the components of these three forces in the directions normal (upwards) to the plane and directly down the plane.

[*Solutions on page 38*]

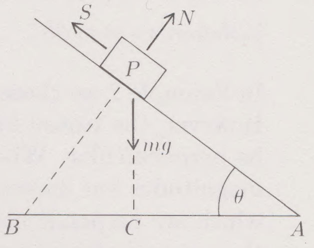


Figure 7

1.2 Static particles

In *Unit 15* you saw how to find the total force vector on a particle by adding together the individual force vectors. You also saw that, by Newton's second law, the total force $\mathbf{F}$ on a static (permanently at rest) particle is zero. This means that the component of the total force in any direction is zero, since any component of the zero vector has the value zero.

In *Unit 24*, being in a static position was described as being in an *equilibrium* position (with no net force acting on the particle).

Example 2

A brick of mass m rests on a plane which is inclined at an angle θ to the horizontal. The coefficient of static friction between the brick and the plane is $\mu = 1$. What is the maximum value of θ for which the brick can remain static?

Solution

The forces acting on the brick are the force of gravity, the normal reaction and the force of friction, as shown in Figure 7 above. As the brick is in equilibrium, the total force, and hence its component in any direction, is zero. To find the component of the total force in some direction, we sum the components of the individual forces in that direction. On resolving the forces in the direction normal to the plane (using the results of Exercise 3), we have

$$N - mg\cos\theta + 0 = 0 \quad \text{or} \quad N = mg\cos\theta.$$

Resolving the forces down the plane gives

$$mg\sin\theta - S + 0 = 0 \quad \text{or} \quad S = mg\sin\theta.$$

Now from the properties of friction, we know that

Unit 15 Subsection 3.2

$$S \leq \mu N,$$

which, after substituting the above expressions for N and S, gives

$$\tan\theta \leq \mu.$$

As $\mu = 1$, this means that the maximum slope of the plane for which the brick can remain static is $\theta = \frac{1}{4}\pi$. □

In the above example we have repeated Example 3 of *Unit 15* Section 3, by resolving the forces on the particle in two perpendicular directions. We have implicitly taken the x-axis of that example to be the 'down the plane' direction here and the y-axis to be normal to the plane. You may like to compare these two solutions in order to appreciate the benefits for two-dimensional problems of our resolution approach here over the earlier vector approach.

Exercise 4

A particle is in equilibrium under the action of four forces. The magnitude and direction of three of the forces are 4 N due North, 7 N South-East and 4 N South 30° West.

(i) Find the components of the fourth force in directions due North and due East.

(ii) Hence find the magnitude and direction of the fourth force.

Give your answers to two decimal places.

[*Solution on page 38*]

In Example 2 we chose to resolve the forces normal to and parallel to the inclined plane. However, the reason for this choice of directions was not just because they happened to be perpendicular. Whenever a problem features two forces which have unknown magnitudes but known directions, it is convenient to resolve the forces in directions which are perpendicular to one of the unknown forces. This ensures that only one of the unknown forces appears in each equation, as illustrated in the following example.

Example 3

A pendulum is composed of a particle bob of mass m fixed to one end of a light inextensible string, which is suspended from a fixed point O. The bob is pulled to one side by a horizontal force of magnitude P, and remains static with the taut string making an angle α with the downward vertical. Find the magnitude of the horizontal force and the tension in the string, in terms of m, α and the magnitude g of the acceleration due to gravity.

Solution

Let T be the tension in the string. The forces acting on the pendulum bob are as indicated in Figure 8. Resolving the forces vertically (that is, perpendicular to the force of unknown magnitude P) gives

$$T\cos\alpha - mg = 0 \quad \text{or} \quad T = mg/\cos\alpha.$$

Resolving the forces perpendicular to the string (and hence perpendicular to the force of magnitude T), we obtain

$$P\cos\alpha - mg\sin\alpha = 0 \quad \text{or} \quad P = mg\tan\alpha. \quad \square$$

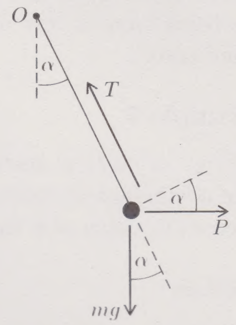


Figure 8

The procedure developed for two-dimensional problems involving a particle in equilibrium is summarized below.

Solution of two-dimensional static particle problems

1. Draw a diagram indicating all of the forces acting on the particle.
2. Resolve each of the forces in a chosen direction.
3. Equate to zero the sum of these components.
4. Repeat Steps 2 and 3 for another direction (which may be at right angles to the first).

Exercise 5

A static particle of mass 1 kg is suspended by two light inextensible strings from two points at the same horizontal level. The strings make angles of $\frac{1}{6}\pi$ and $\frac{1}{4}\pi$ with the horizontal. Find the tensions in the strings.

Exercise 6

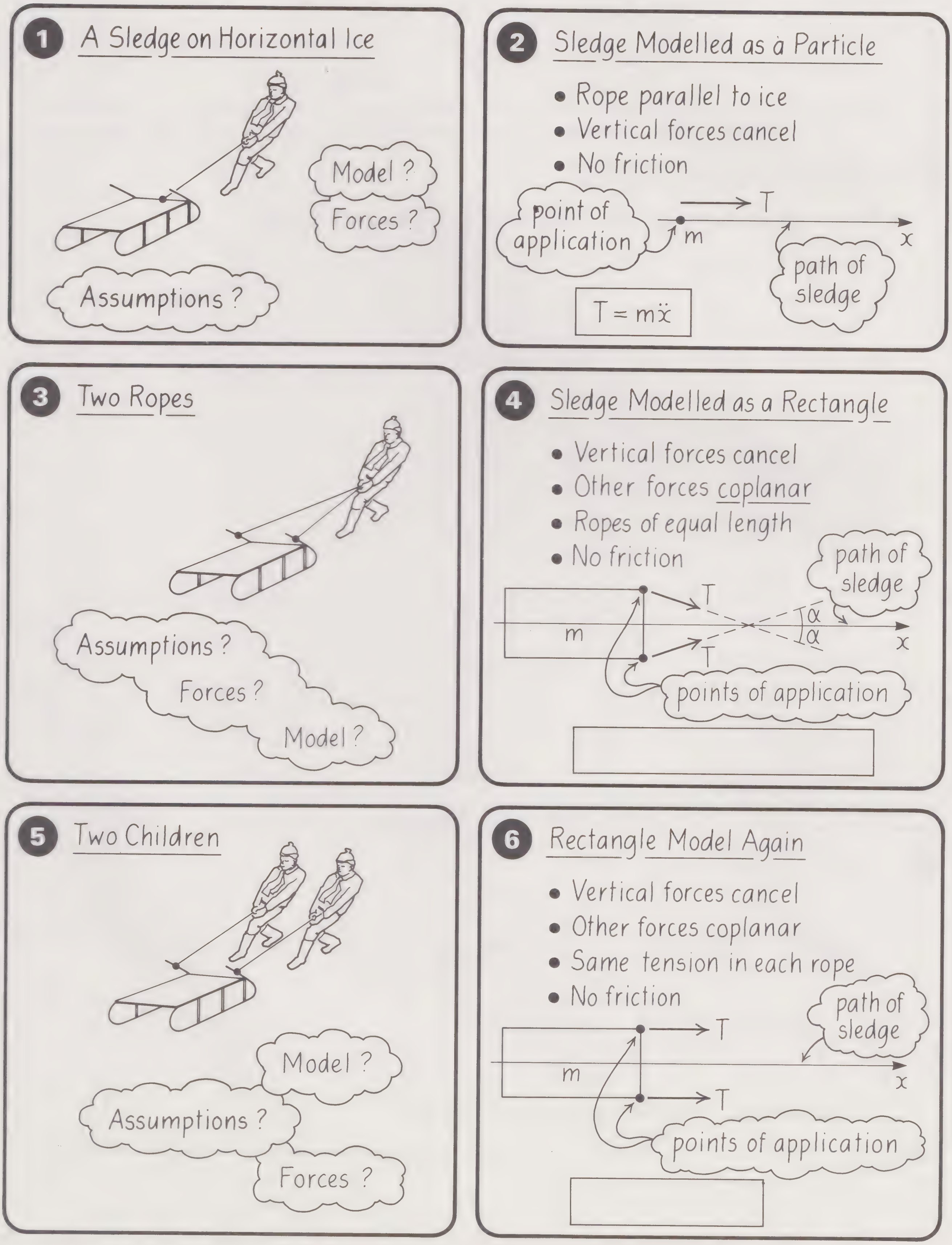
The ends of a light inextensible string of length l are fastened to two points A and B at the same horizontal level and at a distance a apart. A smooth ring of mass m is capable of sliding on the string, but when a horizontal force of magnitude P is applied to the ring, it rests in a static position vertically below B. Show that $P = mga/l$, and that the tension in the string is $mg(l^2 + a^2)/(2l^2)$.

[*Solutions on page 38*]

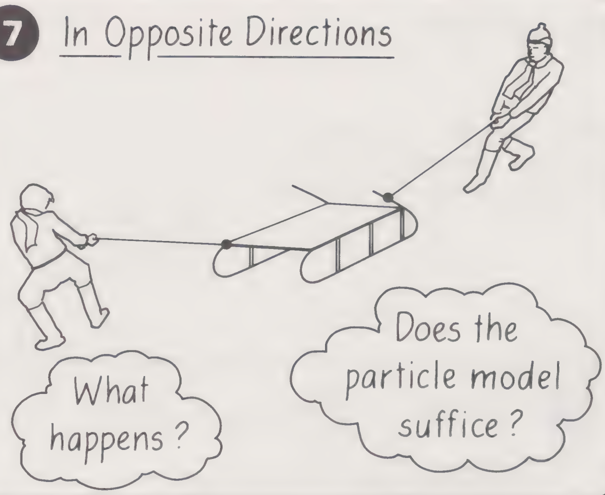
1.3 From particles to extended objects (Audio-tape Subsection)

In the earlier part of the course we modelled objects by using particles. As already pointed out, in adopting this model we have neglected any rotational motion of the object. On the audio-tape we begin to explore the type of model which is required in order to take account of the turning effect of forces.

Start the audio-tape when you are ready.

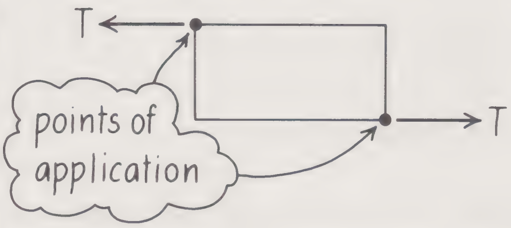


7 In Opposite Directions



8 Zero Total Force

- Assumptions as in Frame 6

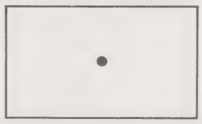


Total force = $\underline{0}$, but...
sledge rotates!

9 Homogeneous Rigid Objects

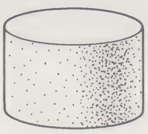
Three-dimensional objects represented in two dimensions:


rectangular block

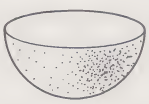

rectangle

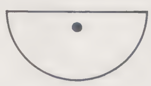

spherical ball


circle


cylindrical drum


circle

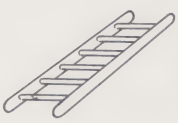
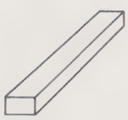
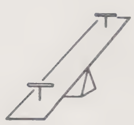

hemispherical bowl


semicircle

Each object has an axis of symmetry (which includes centre of mass)

For block, ball and drum, centre of mass is at geometric centre

10 Uniform Rods

 ladder or  beam or  see-saw

represented by a  line segment

'homogeneous linear object' ≡ 'uniform rod'

In the next section we shall consider not just the magnitudes and directions of forces, but also where on a body the forces act. When an extended body is subjected to several forces, it is possible to treat the body as a particle and as extended in turn. The particle view suffices to evaluate the total force acting on the body, which may be done as in Subsection 1.2 by forming the sums of force components in particular directions. On treating the body as extended, it is possible to consider the points at which the individual forces act in terms of the geometry of the body. You will see that the conditions required for an extended object to be static are similar in nature to the corresponding conditions for a static particle, but in the extended case an extra condition is needed to ensure that the object remains static.

Summary of Section 1

1. The component of a force $\mathbf{F}$ in a direction making an angle α with that of the force is equal to $F \cos\alpha$, where $F = |\mathbf{F}|$.
2. If a particle is static (in equilibrium), then the sum of the components of the forces acting on the particle in any direction is equal to zero.

2 The moment of a force

A particle can undergo translational motion only. This means that it can move from one position to another, but it cannot rotate or spin. Rotation of a particle is ruled out because it has no internal structure, consisting by definition of a single point, whereas the rotation of a rigid object is characterized by relative motion between distinct points of the object. A rod, for example, can undergo rotational as well as translational motion. The conditions for the equilibrium of an extended body, such as a rigid rod, must therefore reflect the fact that it is not turning, as well as fulfilling the conditions of the previous section which arise from the absence of translational motion.

In this section we introduce the idea of an extended body turning or rotating about a pivot point, considering how the turning effect of the forces which act on the object may be described mathematically. We shall ask you to accept, on the basis of some simple experiments but without much mathematical justification at this stage, the conditions which the forces have to satisfy when an extended object remains at rest. You will then be able to tackle problems involving static extended bodies, and not just static particles. Subsection 2.3 contains several such problems.

These conditions will be derived in *Unit 29*.

2.1 The turning effect of a force

Suppose that you try to balance a ruler on a horizontally extended finger. Then you are in effect trying to locate the ruler's centre of mass, for unless this point is directly above your finger then the gravitational force will cause the ruler to turn about your finger. Turning occurs if the vertical line of action of the force (through the centre of mass) is at a non-zero distance from the pivot (your extended finger).

Verbal and pictorial descriptions of objects in balance or turning are all very well, but we are aiming for a mathematical interpretation of what is meant by 'turning'. The size of the turning effect of a force needs to be quantified in terms of what is already known, namely, the magnitude and direction of the force, and the point of the object at which the force acts.

In order to investigate this question, you are asked to perform a simple experiment. Balance a 30 cm ruler on an eraser, the edge of your calculator box or some other object that is not too wide, so that this 'pivot' is a short height above the table surface at which you are working. Place two 1p coins either side of the pivot, so that each is

10 cm from the pivot. Then experiment with moving one of the coins in steps of 2 cm from its initial position, and see how the other coin has to be moved in order to re-establish balance. The conclusion from this activity is not, perhaps, a surprising one: coins of equal weight have to be placed at equal distances on either side of the pivot if the ruler is to remain in balance.

Next place two 1p coins together at a point on the ruler, say at 6 cm from the pivot. Where does a third 1p coin have to be placed to achieve balance? You should find that two identical coins placed together at 6 cm from the pivot are balanced by another identical coin placed on the other half of the ruler at 12 cm from the pivot. If you continue to experiment with varying numbers of 1p coins placed at various pairs of positions along the ruler, you will find in each case that if the weights of the two coin collections are unequal then, in order to achieve balance, the greater weight has to be placed nearer to the pivot than the smaller weight. The turning effect due to the pennies acting at a point depends not only on the weight of the pennies but also on the distance of the point of action from the pivot. On the basis of your experiments, you should find the following result fairly plausible.

A horizontal uniform rod, pivoted at its centre of mass G, will remain horizontal under the action of two forces of magnitudes F_1 and F_2 acting vertically downwards at distances l_1 and l_2 respectively on either side of the pivot (see Figure 1), provided that

$$F_1 l_1 = F_2 l_2.$$

In other words, the distances of the forces from the pivot are in inverse ratio to the magnitudes of the forces.

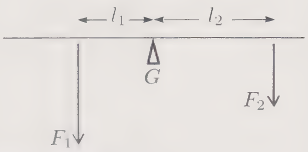


Figure 1

Exercise 1

Jill's weight is $\frac{5}{6}$ that of Jack, who is sitting a distance l from the centre of a uniform see-saw. If the see-saw is at rest and horizontal, where is Jill sitting?

[*Solution on page 39*]

In the situation described in Exercise 1, Jill is unable to affect the balance of the see-saw by altering her weight, but she can do so by changing her position relative to the pivot. If she were to sit at the same distance from the pivot as Jack then Jack's end of the see-saw would sink, whereas hers would rise. Alternatively, if Jill moved further away from the pivot than her position for balance, then her end of the see-saw would sink.

Figure 2 illustrates the situation in Jill's half of the see-saw. (The forces due to the see-saw's mass and the reaction at the pivot are not shown.) In Figure 2(a) Jill's distance d from the pivot, as calculated in Exercise 1, is such that the turning effect of her weight W just balances the turning effect of Jack's weight. Figure 2(b) illustrates what happens when Jill is at a distance less than d from the pivot. The turning effect due to Jill's weight still acts clockwise about the pivot, but it is not large enough to stop the see-saw from turning anticlockwise. For the same weight, the turning effect is smaller, because the distance from the pivot has been decreased. Similarly, Figure 2(c) shows that the turning effect due to Jill's weight is increased if she moves further away from the pivot, and this will result in the see-saw turning clockwise.

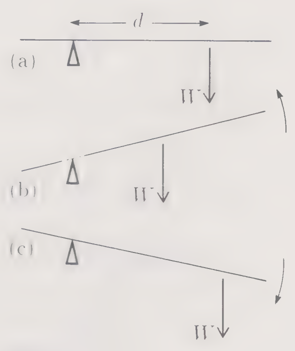


Figure 2

Now suppose that Jane, whose weight is different from Jill's, takes Jill's place on the see-saw at a distance d from the pivot. If Jane's weight is greater than Jill's then the turning effect will be increased, and the see-saw will turn clockwise. On the other hand, if Jane is lighter then the turning effect will be smaller, and the see-saw will turn anticlockwise.

There are many instances in everyday life where the above ideas are put to practical use. For example, suppose that you wish to open a can of paint. You need to exert quite a large force to lift the lid of the can, and this can be done by applying the 'law of levers', using a long-handled object. The edge of the can acts as a pivot, and the large upward force at the edge of the lid (a small distance from the pivot) is supplied by exerting a smaller downward force at a greater distance from the pivot, near the other end of the lever.

2.2 A mathematical description of the turning effect

In this subsection we quantify the turning effect of a force, by defining a quantity Γ called the *moment* or *torque* of a force. These names are synonymous, but in the current context of two-dimensional systems we shall use the word 'moment'.

The symbol Γ is the upper-case Greek letter gamma.

The word 'torque' will be used in *Unit 29*, in the context of three-dimensional systems, and later in this unit.

The examples in the previous subsection are typical of many which suggest that the turning effect of a force may be made larger by increasing either the magnitude of the force or the distance from the force's line of application to the pivot. The following definition is consistent with these observations.

> The **moment** Γ of a force about a fixed point O is the product of the magnitude F of the force and the perpendicular distance d from the point O to the line of action of the force, that is,
>
> $$\Gamma = Fd \tag{1}$$
>
> (see Figure 3).

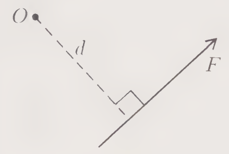


Figure 3

In terms of the see-saw example of the previous subsection, it follows from this definition that the see-saw balances when the moment of Jill's weight is equal to the moment of Jack's weight.

The SI units for moment are N m, or equivalently kg m^2 s^{-2}.

Now force is a vector quantity, and you may therefore be wondering whether the same is true of the moment of a force. Actually moment *is* a vector quantity, and the definition above relates only to the *magnitude* of the moment. For the time being we shall treat the directional aspect simply by specifying whether the moment acts in a clockwise or anticlockwise sense when looking down on the page. For consistency with the later vector definition of moment or torque, the above definition should begin

> The magnitude Γ of the moment of a force ...,

but for this section the above definition will suffice. The vector nature of the moment of a force will be discussed in detail in Subsection 4.1.

You will have learnt by experience that when closing a heavy door it is easier to push near the handle edge rather than close to the hinge. Furthermore, it is more effective to push at right angles to the plane of door rather than along this plane. This experience is borne out by considering the moments about the door's hinge of the forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ and $\mathbf{F}_4$ shown in the plan view of Figure 4. These four forces have different lines of action, but we assume that they have equal magnitude F. The points of application of the three forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ are all at the handle edge of the door, a distance d from the hinge. However, the line of action of the force $\mathbf{F}_1$ is in the plane of the door. This line passes through the hinge and so, using Equation (1), the moment Γ_1 of the force $\mathbf{F}_1$ about the hinge is zero. In contrast, the moment Γ_2 of the force $\mathbf{F}_2$, which is perpendicular to the plane of the door, is $\Gamma_2 = Fd$. The line of action of the oblique force $\mathbf{F}_3$ is at a perpendicular distance $d \sin\theta$ from the hinge, and so the moment of this force about the hinge is $\Gamma_3 = Fd\sin\theta$. This reduces to the previous results for $\theta = 0$ and for $\theta = \frac{1}{2}\pi$, but it also verifies that the maximum turning effect *is* obtained by taking $\theta = \frac{1}{2}\pi$, that is, by pushing perpendicular to the plane of the door. For the fourth force, $\mathbf{F}_4$, the distance between the line of action and the hinge is less than that for $\mathbf{F}_2$, and so $\Gamma_4 < \Gamma_2$. Summarizing, to obtain the maximum turning effect, we should push the door at the handle edge in a direction perpendicular to the plane of the door.

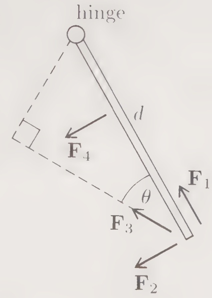


Figure 4

Let us now return to the see-saw example of the previous subsection. The see-saw is supposed uniform and pivoted at its centre of mass, so the only moments that arise are from the gravitational forces on objects or children positioned on the see-saw at non-zero distances from the pivot. (The gravitational force due to the see-saw's mass and the reaction force at the pivot both have zero moment about the pivot, because their lines of action pass through the pivot.) If two objects are placed on either side of the pivot and the see-saw is balanced, then the moment due to one object, acting clockwise, cancels out or balances the moment due to the other object, acting in the opposite sense. A convenient convention is to label the moment of a force with a superscript plus sign if it tends to turn the body on which the force acts in an anticlockwise sense (when looking down on the page), and to label it with a superscript

minus sign if it tends to turn the body in a clockwise direction. So if Γ^+ is the magnitude of the anticlockwise moment due to one object on the see-saw and Γ^- is the magnitude of the clockwise moment due to the other object then, for balance, we have

$$\Gamma^+ = \Gamma^- \quad \text{or} \quad \Gamma^+ - \Gamma^- = 0.$$

If several objects are placed on one arm of the see-saw and it is balanced horizontally by other objects placed on the opposite arm, then the total turning effect in one sense, anticlockwise say, about the pivot must balance the total turning effect in the opposite sense. The see-saw is a special case of a rod, pivoted at a point (its centre of mass) and subjected to forces acting about the pivot in such a way that it remains in equilibrium. The sum of all the anticlockwise moments balances the sum of all the clockwise moments. In symbols, this may be written as

$$\sum_i \Gamma_i^+ = \sum_j \Gamma_j^-$$

or

$$\sum_i \Gamma_i^+ - \sum_j \Gamma_j^- = 0. \qquad (2)$$

Exercise 2

A rigid, uniform, light beam of length 3 m is supported on a cross-member (the pivot) 1 m from one end. The beam has a mass of 50 kg suspended from the end of the longer arm. What vertical force must act at the end of the shorter arm for the beam to balance in the horizontal position?

[*Solution on page 39*]

So far we have considered the equilibrium of see-saws, rods and beams whose pivot is at their centre of mass (when the mass is non-negligible). If the pivot is at another point of the rod, then we need also to consider the turning effect caused by the weight of the rod itself. As you will see in Section 4, this moment may be calculated by considering the weight of the rod to be concentrated at its centre of mass. For this reason the centre of mass is often called the *centre of gravity*.

Example 1

Suppose in Exercise 2 that the beam is no longer considered as 'light', and that its mass of 10 kg is to be taken into account. Calculate the vertical force which needs to be applied at the end of the shorter arm in this case, to achieve horizontal balance.

Solution

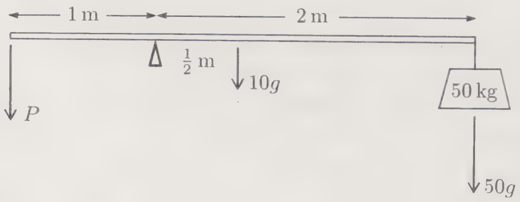


Figure 5

Let the magnitude of the required force be P N, as shown in Figure 5. The anticlockwise moment about the pivot is, as before,

$$\Gamma^+ = P \times 1 = P.$$

The weight of the beam produces a clockwise moment, and this moment may be calculated by assuming that the weight of the beam acts at the centre of mass, as indicated in Figure 5. So the moment due to the weight of the beam is

$$\Gamma_1^- = 10g \times 0.5 = 5g,$$

while the clockwise moment caused by the 50 kg mass is

$$\Gamma_2^- = 50g \times 2 = 100g.$$

From Equation (2), the condition for the beam to be in equilibrium is

$$\Gamma^+ - (\Gamma_1^- + \Gamma_2^-) = 0,$$

which leads directly to

$$P = 105g.$$

In other words, the required force is $105g$ N vertically downwards at the end of the shorter arm. (This magnitude is slightly larger than the answer to Exercise 2, as expected.) □

Exercise 3

A plank, which is 6 m long and weighs 500 N, projects by $2\frac{1}{2}$ m over the side of a quay. What weight must be placed on the landward end of the plank so that a person weighing 600 N may walk to the other end without the plank tipping over? (Consider moments about the pivotal point at the edge of the quay.)

[*Solution on page 39*]

2.3 Static rigid bodies

The examples of the last subsection were all concerned with the equilibrium of objects which were able to turn about a fixed pivot. In these cases we were able to use the principle that the total moment *about the pivot* of the external forces acting on the body was zero, where by **total moment** we mean the sum of the anticlockwise moments minus the sum of the clockwise moments. In this subsection we consider the more general case of the equilibrium of objects which need not have a pivot, such as a ladder leaning against a wall. From *Unit 17*, you already know that the total external force on such a static object is zero. But now we can extend the result of the previous subsection, and use the principle that the total moment about *any fixed point* of the body is zero. At this stage we shall use this important result without proof. In symbols, the statement of this extended principle appears identical to the version given in Subsection 2.2, namely

Unit 17 Section 2

$$\sum_i \Gamma_i^+ - \sum_j \Gamma_j^- = 0. \tag{2}$$

Here Γ_i^+ is the magnitude of the ith anticlockwise moment about a point due to the corresponding force on a static rigid body, in a particular plane. Similarly, Γ_j^- is the jth clockwise moment in the same plane. The sum of the total turning effect in one sense balances or cancels out the total turning effect in the opposite sense. The difference between the current application of Equation (2) and that used in the previous subsection is that now the moments may be calculated about any chosen point, and not just about a pivot.

Just as objects on which there is no external force can be in translational motion (although, as you saw in *Unit 17*, the centre of mass of the object will then be moving with constant velocity), objects on which there is no total moment may be rotating. An example of this is a spinning satellite in deep space.

The procedure used in Subsection 1.2 to solve static particle problems can now be extended to cover static extended-body problems.

Solution of two-dimensional static rigid-body problems

1. Draw a diagram indicating all of the external forces acting on the body.
2. Resolve each of the external forces in a chosen direction.
3. Equate to zero the sum of these components.
4. Repeat Steps 2 and 3 for another direction (which may be at right angles to the first).
5. Calculate the moment of each external force about a chosen point, and classify each moment as anticlockwise or clockwise about that point.
6. Equate to zero the total moment, which is the sum of the anticlockwise moments minus the sum of the clockwise moments.

In saying that a body is *rigid*, we mean that the distance between each pair of particles within the body is fixed.

Example 2

A uniform beam, 3 m long and of mass 40 kg, is supported horizontally by two vertical strings. Each of the strings is attached at a distance 0.5 m from an end of the beam. What weight (in newtons) should be placed on the beam at one end, so that the tension in one of the strings just vanishes?

Solution

Suppose that the tensions in the strings are T_1 N and T_2 N, and that the weight of magnitude W N is placed at the right-hand end of the beam (see Figure 6).

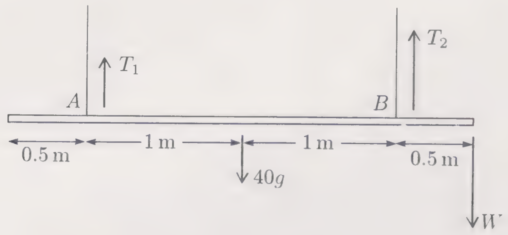


Figure 6

The weight will tend to turn the beam in a clockwise direction, and so it is the left-hand string which will become slack, that is, $T_1 = 0$. (If you do not accept this argument, you can show that the alternative assumption $T_2 = 0$ leads to T_1 being negative, which is physically impossible.)

Consider the balance of moments about the point B of attachment of the right-hand string. The only anticlockwise moment is caused by the weight of the beam, so

$$\Gamma^+ = 40g \times 1 = 40g.$$

As $T_1 = 0$, the only clockwise moment is due to the unknown weight W. Hence

$$\Gamma^- = W \times 0.5 = \tfrac{1}{2}W.$$

As the total moment about B must be zero, we have

$$40g - \tfrac{1}{2}W = 0 \quad \text{or} \quad W = 80g.$$

Hence a weight of $80g$ N placed at one end of the beam will cause the tension in the further string to be zero.

Notice that we have not used resolution of forces at all in this example. None of the forces have horizontal components. The resolution of the forces in a vertical direction would give the value of T_2 (in fact, $T_2 = 120g$), but this was not required. □

When reading the solution to the above example, you may have wondered if we would have obtained a different result upon choosing another point, such as A, about which to consider the balance of moments. The answer is 'no' (as illustrated in Exercise 4 below), but a judicious choice of the point about which moments are to be taken can greatly simplify the solution. The moment of a force about a point on that force's line of action is zero. So if we have a force of unknown magnitude and we take moments about a point on its line of action, then it will not contribute to the equation. This consideration motivated the choice of the point B in Example 2.

Exercise 4

Repeat Example 2, taking moments about the centre of mass of the beam.

Exercise 5

Consider again the static beam in Example 2, but suppose now that instead of the tension in one string just vanishing, the tension in one string is double that in the other string. What weight must be placed on one end of the beam to achieve this state of affairs?

[*Solutions on page 39*]

The above example and exercises concerned systems where all the forces act in a vertical direction. The procedure for static rigid-body problems is equally applicable to more general cases, although it may then be necessary to resolve the forces in two distinct directions.

Example 3

A uniform ladder of mass M and length l rests with its top against a smooth vertical wall and its base on rough horizontal ground. If the inclination of the ladder to the horizontal is θ, find the magnitudes and directions of the external forces acting on the ladder.

Solution

The forces acting on the ladder are as shown in Figure 7. Since the ladder is uniform, its weight, of magnitude Mg, acts vertically downwards through its mid-point. The vertical wall is smooth, and so we assume that there is no frictional force where the ladder touches the wall, but there is a normal reaction here of magnitude S. At the base of the ladder there is a frictional force of magnitude T and a normal reaction of magnitude R, as shown in the diagram. The direction of the frictional force has been chosen as shown because experience indicates that if friction is overcome then the bottom of the ladder slips away from the wall. (The opposite choice would lead to T being found to be negative.)

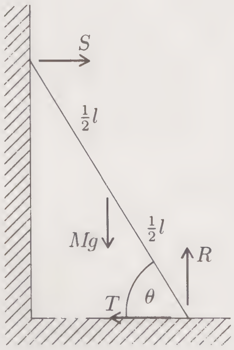


Figure 7

Taking moments about the bottom of the ladder (which eliminates two of the unknown quantities) gives

$$S \times l \sin\theta - Mg \times \tfrac{1}{2} l \cos\theta = 0,$$

which leads to

$$S = \tfrac{1}{2} Mg \cot\theta.$$

Resolving the forces vertically gives

$$R = Mg,$$

whereas resolving the forces horizontally gives

$$T = S = \tfrac{1}{2} Mg \cot\theta. \quad \square$$

Exercise 6

A uniform ladder of mass M and length l stands on rough horizontal ground and rests against a smooth vertical wall. Find the minimum angle, θ, between the ladder and the horizontal for which the ladder can remain static, assuming that the coefficient of static friction between the base of the ladder and the ground is $\mu = 1$.

Exercise 7

A uniform rod, of mass m and length $2a$, rests with its lower end in contact with the inside of a smooth hemispherical bowl of radius a, whose axis is vertical. The upper end of the rod projects beyond the rim of the bowl. Show that the inclination θ of the rod to the horizontal satisfies the equation

$$2\cos 2\theta = \cos\theta,$$

and hence find the angle θ. (You may find the identity $\cos 2\theta = 2\cos^2\theta - 1$ useful.)

[*Solutions on page 40*]

As you will have observed already, it is important to draw a diagram at the beginning of each problem and to identify clearly the forces and their points and lines of action. When tackling problems in mechanics, it is essential to extract the key words from the problem description. For example, the procedure above can be used only if the key words 'static' or 'at rest', or their synonyms, appear in the description of the problem. Mechanics has developed its own vocabulary, as a shorthand for the mathematical models of the more complicated physical systems which it describes. You are already familiar with such epithets as 'light' and 'inextensible' (for a string) meaning that the mass may be neglected and that the length is constant. If a body or the surface with which it is in contact is 'smooth', then we can neglect the frictional force at the point of contact. The reaction force will, as usual, be normal to the surface at the point of contact.

A body may be described as being 'smoothly hinged at' or 'freely jointed at' or 'free to turn about' a point P, meaning that it is valid to neglect the frictional forces close to P and thus the moments due to these forces. In general there will be a reaction force at P, but its direction will not usually be known before the calculation commences. It is a good idea therefore to take moments about P, so as to eliminate this unknown reaction. The following example illustrates this.

Example 4

A uniform beam AB, of length l and mass m, is free to turn in a vertical plane about a smooth hinge at A. It is supported in a horizontal, static position by a light string attached to the end B of the beam and to a point C at a height h vertically above A. Find the tension in the string.

Solution

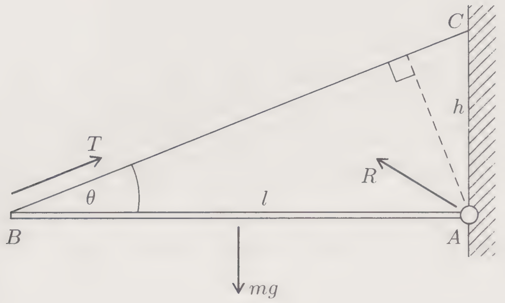


Figure 8

The forces acting on the beam are as shown in Figure 8. The string makes an angle θ with the horizontal. Since the beam is uniform, its weight, of magnitude mg, acts vertically downwards at the mid-point. The direction of the reaction of magnitude R at the hinge is unknown, so to find the tension T in the string, we take moments about the hinge at A. This gives

$$mg \times \tfrac{1}{2}l - T \times l \sin\theta = 0.$$

Hence we have

$$T = \frac{mg}{2\sin\theta} = \frac{mg\sqrt{l^2+h^2}}{2h}. \quad \square$$

Exercise 8

Suppose that, from the situation described in Example 4 and shown in Figure 8, the end B of the beam is raised by shortening the string until BC is horizontal. With the system in this new position, find the tension in the string.

[*Solution on page 40*]

Summary of Section 2

1. The magnitude Γ of the **moment** of a force about a fixed point O is the product of the magnitude F of the force and the perpendicular distance d from the point O to the line of action of the force, that is,

   $$\Gamma = Fd.$$

2. The **total moment** of the forces acting in two dimensions on a body about any point is the sum of the anticlockwise moments minus the sum of the clockwise moments.

3. If a rigid body is static (in equilibrium), then both the total force on the body and the total moment of forces acting on the body about any point are zero.

3 Motion in a circle

Following the introduction of Newton's second law in *Unit 4*, we dealt with problems involving the linear motion of a particle. We shall now pursue a similar course for particles which undergo circular motion, which is a special type of two-dimensional motion. As before, the analysis is based on the use of Newton's second law to relate the acceleration of a particle to the forces which cause that acceleration, and the outcome is the appropriate equation of motion for the particle.

Circular motion was investigated previously, in *Unit 15* Section 5, for the particular case of a pendulum. The analysis here will not assume any familiarity with that material, since the current approach is both more general and rather different in kind. Instead of using mainly Cartesian coordinates, as in *Unit 15*, we shall put more emphasis on employing plane polar coordinates, because these are better suited for the purpose of studying circular motion.

Before considering the forces which cause motion in a circle, we concentrate on the kinematics of such motion. This involves the development of formulas to describe the position, velocity and acceleration of a particle moving in a circle.

3.1 Kinematics of circular motion

Suppose that a particle P moves in a circle. The position of this particle may be given either in terms of its Cartesian or polar coordinates with respect to an origin at the circle's centre, or as a position vector. Our aim is to use these descriptions of position, and the fact that the path of the particle is a circle, to derive expressions for its velocity and acceleration in as simple a form as possible.

With respect to a set of fixed Cartesian axes in the plane, the particle's position at any time t can be described by its Cartesian coordinates (x, y), as shown in Figure 1. An alternative description of the position of the particle is in terms of its position vector, $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$, which is a displacement vector drawn from the origin O to the particle P. If the particle is moving on a circle with centre O, then we know that the magnitude of $\mathbf{r}$ remains fixed (and is non-zero) but that its direction, measured as the anticlockwise angle θ from the positive direction of the x-axis, is changing. Indeed, another description of the particle's position at any instant of time is in terms of its polar coordinates $[r, \theta]$, as described in *Unit 5* Subsection 3.1. The equations connecting these three descriptions of position are

Unit 15 Subsection 1.2

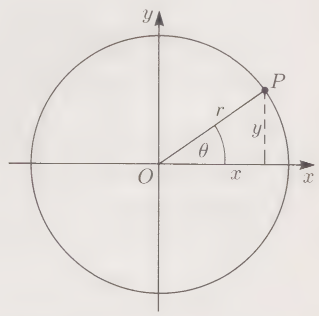


Figure 1

$$x(t) = r\cos\theta(t), \tag{1}$$
$$y(t) = r\sin\theta(t), \tag{2}$$

linking the particle's Cartesian and polar coordinates (see Figure 1), and

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}, \tag{3}$$
$$\mathbf{r}(t) = r\cos\theta(t)\,\mathbf{i} + r\sin\theta(t)\,\mathbf{j}, \tag{4}$$

which link the position vector to the Cartesian and polar coordinates of the particle, respectively. Here $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the respective directions of the Cartesian x- and y-axes.

In Equations (1)–(4) the dependence upon time has been indicated explicitly. Notice that, since the particle remains on a circle, the length of OP does not change with time, so that r does not depend on t. Thus of the right-hand sides of Equations (3) and (4), the latter is the more natural to use in describing circular motion, since here only one of the independent variables, θ, is varying with time. However, because $\mathbf{r}$ is a function of θ, which is in turn a function of t, the chain rule must be employed in any differentiation involving t. This is demonstrated in the following exercise.

Exercise 1

By differentiating Equation (4) with respect to t, find an expression for the velocity vector $\dot{\mathbf{r}}$ in terms of r, θ, $\dot{\theta}$, $\mathbf{i}$ and $\mathbf{j}$. (Notice that the unit vectors $\mathbf{i}$ and $\mathbf{j}$ do not vary with time.)

[*Solution on page 40*]

Since the particle is moving in the plane, its velocity $\dot{\mathbf{r}}$ may be expressed in terms of its components $\dot{x}$ and $\dot{y}$ with respect to the Cartesian coordinate system, as explained in *Unit 15* Subsection 1.3. There you saw that the velocity vector is given by

$$\dot{\mathbf{r}} = \dot{x}\,\mathbf{i} + \dot{y}\,\mathbf{j}, \tag{5}$$

which is the outcome of differentiating Equation (3) with respect to time. Hence, using the results of Exercise 1 together with Equation (5), we have

$$\dot{x} = -r\dot{\theta}\sin\theta \tag{6}$$

and $$\dot{y} = r\dot{\theta}\cos\theta. \tag{7}$$

Here each of $\dot{x}$, $\dot{y}$, θ and $\dot{\theta}$ is a function of t, but this has not been indicated explicitly.

From Equation (5), the particle has a velocity vector which is the sum of a vector $\dot{x}\,\mathbf{i}$ in the direction of the x-axis and a vector $\dot{y}\,\mathbf{j}$ in the direction of the y-axis. Furthermore, from Equations (6) and (7) we have expressions for these vectors in terms of polar coordinates. Suppose that we now adopt the approach of Section 1. There we were concerned not with velocities but with (vector) forces, finding by resolution how much of a force acts in specified directions (which were not necessarily parallel to the Cartesian axes). Now resolution into components is not restricted to forces; it is a process which may be applied to any vector, and in any specified direction.

In the case of a particle P moving in a circle, there are two directions of primary interest. These are (see Figure 2):

(i) the radial direction PN, that is, the direction radially outwards from the centre O of the circle towards the particle P (and in the sense of increasing r);

(ii) the transverse direction PT, that is, the direction tangential to the circle (and in the sense of increasing θ).

The components of a vector in these two directions are called the *radial* and *transverse* components, respectively.

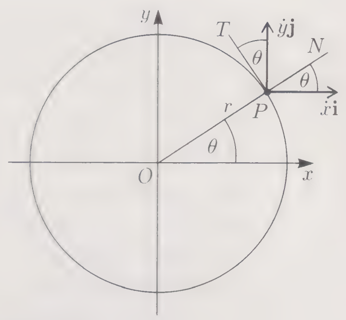


Figure 2

Exercise 2

(i) Find the radial and transverse components of the vector $\dot{x}\,\mathbf{i}$ (see Figure 2).

(ii) Find the radial and transverse components of the vector $\dot{y}\,\mathbf{j}$.

(iii) Hence find the radial and transverse components of the velocity vector $\dot{\mathbf{r}} = \dot{x}\,\mathbf{i} + \dot{y}\,\mathbf{j}$.

[*Solution on page 40*]

In Exercise 2 you found expressions for the radial and transverse components of velocity in terms of $\dot{x}$, $\dot{y}$ and θ. We can now use Equations (6) and (7) to find expressions for these components purely in terms of the polar coordinates r and θ. In this way, we find that the *radial component of velocity* is

$$\begin{aligned}\dot{x}\cos\theta + \dot{y}\sin\theta &= (-r\dot{\theta}\sin\theta)\cos\theta + (r\dot{\theta}\cos\theta)\sin\theta \\ &= 0.\end{aligned} \tag{8}$$

You should not find this result too surprising. In *Unit 14* Subsection 4.5 it was shown that the velocity vector of a particle is always directed along the tangent to the particle's path. In the case of circular motion this means that the velocity is always tangential to the circle, and hence that the component in the radial direction must be zero.

The method used in deriving Equation (8) also allows us to show that the *transverse component of velocity* is

$$\begin{aligned}-\dot{x}\sin\theta + \dot{y}\cos\theta &= -(-r\dot{\theta}\sin\theta)\sin\theta + (r\dot{\theta}\cos\theta)\cos\theta \\ &= r\dot{\theta}(\sin^2\theta + \cos^2\theta) \\ &= r\dot{\theta}.\end{aligned} \tag{9}$$

Equations (8) and (9) are the expressions in polar coordinates giving the radial and transverse components for the velocity of a particle moving in a circle (see Figure 3). We shall now follow a similar procedure to derive the components of the particle's acceleration

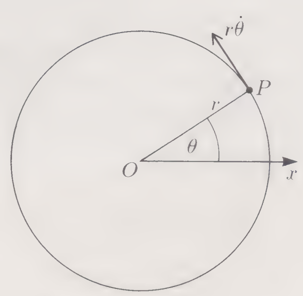


Figure 3

$$\ddot{\mathbf{r}} = \ddot{x}\,\mathbf{i} + \ddot{y}\,\mathbf{j}. \tag{10}$$

In Exercise 1 you found that

$$\dot{\mathbf{r}} = -r\dot{\theta}\sin\theta\,\mathbf{i} + r\dot{\theta}\cos\theta\,\mathbf{j}.$$

Differentiating this with respect to time, and remembering that r is a constant, we obtain

$$\begin{aligned}\ddot{\mathbf{r}} &= -r\frac{d}{dt}(\dot{\theta}\sin\theta)\mathbf{i} + r\frac{d}{dt}(\dot{\theta}\cos\theta)\mathbf{j}\\ &= -r\left(\ddot{\theta}\sin\theta + \dot{\theta}\frac{d}{dt}(\sin\theta)\right)\mathbf{i} + r\left(\ddot{\theta}\cos\theta + \dot{\theta}\frac{d}{dt}(\cos\theta)\right)\mathbf{j}\\ &= -r(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)\mathbf{i} + r(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)\mathbf{j},\end{aligned}$$

after using the expressions for $d(\sin\theta)/dt$ and $d(\cos\theta)/dt$ from the solution to Exercise 1. Comparing this last result with Equation (10) gives

$$\ddot{x} = -r(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) \qquad (11)$$

and $$\ddot{y} = r(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta). \qquad (12)$$

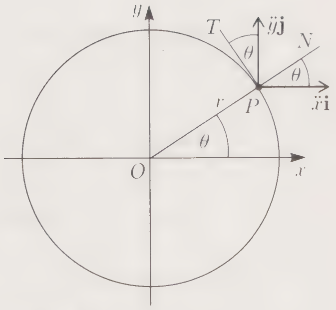


Figure 4

Now the *radial component of the acceleration* $\ddot{\mathbf{r}} = \ddot{x}\,\mathbf{i} + \ddot{y}\,\mathbf{j}$ is (see Figure 4)

$$\begin{aligned}\ddot{x}\cos\theta + \ddot{y}\sin\theta &= -r(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)\cos\theta + r(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)\sin\theta\\ &= -r\dot{\theta}^2(\cos^2\theta + \sin^2\theta)\\ &= -r\dot{\theta}^2, \qquad (13)\end{aligned}$$

and the *transverse component of the acceleration* is

$$\begin{aligned}-\ddot{x}\sin\theta + \ddot{y}\cos\theta &= r(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)\sin\theta + r(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)\cos\theta\\ &= r\ddot{\theta}(\sin^2\theta + \cos^2\theta)\\ &= r\ddot{\theta}. \qquad (14)\end{aligned}$$

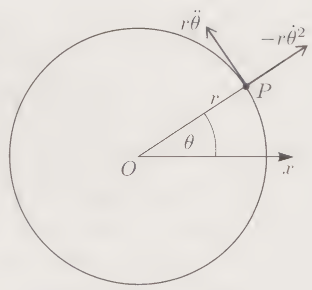


Figure 5

These components are shown in Figure 5. The results which have been derived concerning radial and transverse components are summarized in the box below.

Circular motion

In terms of polar coordinates, the radial and transverse components of the position, velocity and acceleration of a particle moving in a circle, r = constant, are as follows.

Table 1

	Radial component	Transverse component
Position	r	0
Velocity	0	$r\dot{\theta}$
Acceleration	$-r\dot{\theta}^2$	$r\ddot{\theta}$

Exercise 3

A particle is moving on a unit circle centred on the origin O, that is, on the circle $r = 1$. Its polar angle θ at time t is given by $\theta = t^2$. Find the radial and transverse components of the particle's velocity and acceleration at time t.

Exercise 4

A particle moves in a circle of radius r, with transverse velocity component v. Show that

(i) the speed of the particle is $|v|$;

(ii) the radial component of the particle's acceleration is $-v^2/r$.

[*Solutions on page 40*]

You may have noticed, from either Equation (13) or the result of Exercise 4(ii), that the component of acceleration along the outward radial direction is *negative*. The minus sign here indicates that the acceleration of a particle moving in a circle always has a positive component *inwards*, towards the centre of the circle. It was pointed out in *Units 4* and *15* that, when used mathematically, acceleration has a more subtle

meaning than it does in everyday language. In normal usage it tends to mean 'rate of increase in speed', whereas Exercise 4(ii) shows that acceleration in the mathematical sense is also linked with changes in the direction of motion, and may be non-zero even when the speed is constant. The following exercise illustrates the fact that a particle moving with constant speed around a circle has non-zero acceleration.

This point is emphasized in the television programme for this unit (Subsection 4.2).

Exercise 5

A particle is moving in a circle, $r =$ constant, centred on the origin O. Its polar angle θ at time t is given by $\theta = \omega t$, where ω is a constant.

(i) Find the radial and transverse components of the particle's velocity and acceleration.

(ii) Show that the particle's speed $|v|$ is constant, being given by

$$|v| = r|\omega|.$$

Hence show that the time for the particle to complete each revolution of the circle is

$$\tau = 2\pi/|\omega|.$$

[*Solution on page 41*]

The quantity $\dot{\theta} = \omega$ is called the **angular velocity** of the particle, and its magnitude $|\omega|$ is called the **angular speed**. If the particle moves in an anticlockwise direction, then $\dot{\theta} > 0$ and the angular velocity ω is positive. If on the other hand the particle moves clockwise around the circle, then $\dot{\theta} < 0$ and the angular velocity is negative.

Note that here we are discussing two-dimensional motion. In general, angular velocity is a vector quantity, as will be made clear in Section 4.

The transverse velocity component

$$v = r\dot{\theta} = r\omega$$

is similarly positive for anticlockwise motion and negative for clockwise motion. Since the radial component of the velocity is always zero (see Figure 6), we may refer to v as the *velocity* of the particle, provided it is clear from the context that the motion of the particle is confined to a circle. The equation $v = r\dot{\theta}$ is then analogous to the relation $v = \dot{x}$ for the velocity of a particle in one-dimensional linear motion (which may also be positive or negative, depending on the particle's direction of motion).

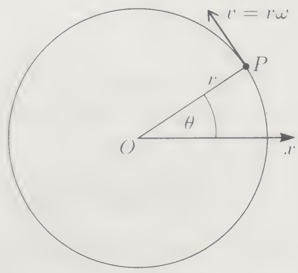


Figure 6

In terms of the angular velocity ω, the acceleration of the particle has radial component $-r\omega^2$ and transverse component $r\dot{\omega}$ (from Table 1 above). The radial component can also be expressed in terms of v as $-v^2/r$. In Exercise 5 you considered the special case of *uniform* circular motion, for which ω is constant. In this case the transverse component of the acceleration is zero (since $\dot{\omega} = 0$), and the acceleration vector of the particle is directed towards the centre of the circle (see Figure 7). The results obtained so far for uniform motion are summarized below.

Uniform circular motion

Consider a particle moving around a circle of radius r with constant angular velocity $\dot{\theta} = \omega$.

(i) The velocity of the particle is $v = r\omega$ transversely, that is, tangential to the circle and in the anticlockwise sense if v, ω are positive (see Figure 6; this result is true even if ω is not constant).

(ii) The acceleration of the particle has magnitude $r\omega^2 = v^2/r$, and is directed towards the centre of the circle (see Figure 7).

(iii) The time taken for one complete revolution of the circle is $\tau = 2\pi/|\omega|$.

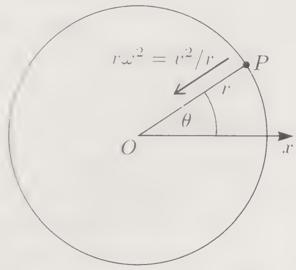


Figure 7

Exercise 6

A fly sits at the tip of the minute hand of a clock. The hand has length 0.5 m and can be assumed to move with constant angular velocity. Find the angular velocity, velocity and acceleration of the fly.

[*Solution on page 41*]

In this subsection we have derived expressions for the components of a particle's acceleration when it is moving in a circle. This acceleration must be caused by forces acting on the particle. For example, the motion of a satellite moving in a circular orbit is caused by the gravitational force which acts on the satellite due to the presence of the Earth, whereas the oscillations of a simple pendulum are caused by the tension in the string and the force of gravity acting on the pendulum's bob. In the rest of this section we shall consider the forces which cause circular motion, considering first the uniform case and then motion in a circle with variable speed.

3.2 Uniform circular motion

If there are no forces acting on a particle, then it is known from Newton's first law that the particle will either remain at rest or move with constant velocity in a straight line. If the particle moves on a circular path, therefore, there must be some force or forces causing it to do so, even when the circular motion is uniform. In the last subsection you saw that a particle moving in a circle of radius r with constant angular velocity ω and velocity v is being accelerated towards the centre of the circle with an acceleration of magnitude $r\omega^2 = v^2/r$. If the particle has mass m, then Newton's second law states that there must be a force on the particle directed towards the circle's centre, of magnitude

$$F = mr\omega^2 = \frac{mv^2}{r}.$$

For uniform circular motion, there is no transverse component of acceleration, and so there is no component of the total force acting on the particle in this direction.

Example 1

Two particles, of equal mass m, are connected by a light inextensible string which passes through a hole in a smooth horizontal table, one particle being on the surface of the table while the other is suspended beneath. The particle on the table describes circles of radius r with a constant angular speed, as a result of which the suspended particle is static. Find the angular speed of the particle on the table.

Solution

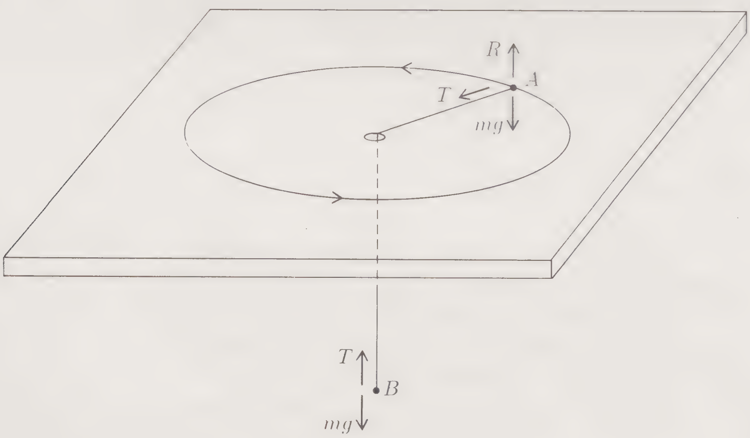


Figure 8

Let the particles be labelled A and B, as shown in Figure 8, and let T be the tension in the string. Consider Particle B. Since this particle is static, the total force acting on it is zero, so that

$$T = mg.$$

The vertical component of Newton's second law applied to Particle A gives

$$R - mg = 0,$$

since the particle remains in the horizontal plane. (However, this equation is not required in solving the current problem.) For uniform circular motion of radius r and angular speed $|\omega|$, the radial component of acceleration is $-r\omega^2$. So the radial component of Newton's second law gives

$$-T = -mr\omega^2.$$

We therefore have

$$\omega^2 = \frac{T}{mr} = \frac{g}{r} \quad \text{or} \quad |\omega| = \sqrt{\frac{g}{r}}. \quad \square$$

Exercise 7

A particle P of mass m moves in a circle on a smooth horizontal table with constant angular velocity ω. It is attached to an end of a light inextensible string of length l. The other end of the string is fixed to a point on the table. What is the tension in the string?

Exercise 8

A particle of mass m lies on a smooth horizontal table and is attached, by a light inextensible string which passes through a hole in the table, to a particle of mass $2m$ suspended below the table. The particle of mass m describes a circle of radius 40 cm on the table with uniform speed $|v|$, so that the particle of mass $2m$ remains at rest. Calculate the speed of the particle on the table.

[*Solutions on page 41*]

The necessary radial component of force required for circular motion may arise from causes other than the tension in a string, as has been the case in the example and exercises above. In the television programme for this unit, the motion of a geostationary satellite is modelled as that of a particle on a circular path around the Earth. There the force connected with the circular motion is the force of attraction on the satellite due to the Earth, which may be calculated according to Newton's universal law of gravitation. In the following exercise, it is a frictional force which holds a particle within its circular path. This can be seen, for example, as a model for a small stone which whirls around while resting on a playground roundabout.

Newton's universal law of gravitation will be discussed in detail in *Unit 30*.

Exercise 9

A plane horizontal disc is constrained to rotate uniformly about its centre, describing two complete revolutions per second. A small object placed on the disc stays at the same point relative to the disc as the latter rotates. The coefficient of static friction between the object and the disc is $\mu = \frac{1}{2}$. Show that the greatest possible distance of the object from the centre of the disc is approximately 3.1 cm.

[*Solution on page 41*]

3.3 Motion in a circle with variable speed

In Example 1 and Exercises 7 and 8, we considered the circular motion of a particle attached to a string, whose motion is confined to a horizontal plane. In this subsection we investigate the circular motion in a vertical plane of a particle attached to a string. You will see that in this situation the speed of the particle does not remain constant, due to the effects of gravity. We might, for example, be modelling the motion of a simple pendulum. This problem was discussed previously, in *Unit 15* Section 5, but its solution is significantly simpler when the equation of motion is expressed in terms of its radial and transverse components.

Suppose that a particle P of mass m moves on a vertical circle of radius r, by swinging at the end of a light inextensible string attached to a fixed point O. For convenience, we measure the inclination θ of the string anticlockwise from the downward vertical, as shown in Figure 9. The forces acting on the particle are:

(i) the force of gravity, of magnitude mg vertically downwards;

(ii) the force due to the tension T in the string, which is directed radially inwards.

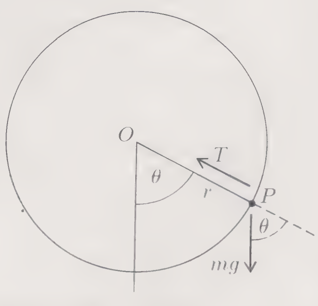


Figure 9

In this case we are not able to assume that the circular motion is uniform; indeed, our physical experience tells us that it will not be! From Table 1, the acceleration of the particle has a radial component $-r\dot{\theta}^2$ (outwards), and a transverse component $r\ddot{\theta}$ (in the direction of increasing θ). So the radial component of the equation of motion is

$$mg\cos\theta - T = -mr\dot{\theta}^2, \tag{15}$$

whereas the transverse component is

$$-mg\sin\theta = mr\ddot{\theta}. \tag{16}$$

We concentrate first on integrating Equation (16), in order to obtain an expression for $\dot{\theta}$. We can then substitute this into Equation (15), to find the tension in the string. Multiplying Equation (16) by $\dot{\theta}/m$, we obtain

$$r\dot{\theta}\ddot{\theta} = -g\dot{\theta}\sin\theta. \tag{17}$$

Now we note that, by the chain rule,

$$\frac{d}{dt}(\dot{\theta}^2) = 2\dot{\theta}\ddot{\theta},$$

This integration 'trick' can be used for all differential equations of the form

$$\frac{d^2y}{dx^2} = f(y).$$

so that Equation (17) can be rewritten as

$$\frac{d}{dt}(\tfrac{1}{2}r\dot{\theta}^2) = -g\sin\theta\frac{d\theta}{dt}.$$

Integrating this equation with respect to time gives

$$\tfrac{1}{2}r\dot{\theta}^2 = -g\int\sin\theta\frac{d\theta}{dt}\,dt = -g\int\sin\theta\,d\theta = g\cos\theta + c.$$

The constant of integration c can be found in terms of the velocity v_0 of the particle at the lowest point of the path, where $\theta = 0$. At this point we have $r\dot{\theta} = v_0$, so that $\dot{\theta} = v_0/r$. This initial condition leads to

$$c = \frac{v_0^2}{2r} - g,$$

from which we deduce that

$$\tfrac{1}{2}r\dot{\theta}^2 = g\cos\theta + \frac{v_0^2}{2r} - g$$

or $$\dot{\theta}^2 = \frac{v_0^2}{r^2} - \frac{2g}{r}(1-\cos\theta). \tag{18}$$

In theory this equation can now be integrated to find θ as a function of time t, but in practice the integral involved cannot be evaluated in terms of elementary functions. However, Equation (18) does permit us to calculate the speed of the particle at any position during its motion.

In the following exercise you are asked to show that Equation (18) is equivalent to the law of conservation of mechanical energy, in the form

Unit 7 Subsection 3.2

kinetic energy + gravitational potential energy = constant.

Energy conservation is often associated with situations where all the forces acting upon a particle depend only on position. Here the tension T in the string depends also on the speed of the particle. Despite this, the law of conservation of mechanical energy still applies, because the direction in which the string force acts is always perpendicular to the direction of motion. This point was explained in detail in *Unit 15* Subsection 5.4.

Exercise 10

Show that, for the model and initial condition discussed above, the law of conservation of mechanical energy in the form

kinetic energy + gravitational potential energy = constant

leads to Equation (18).

[*Solution on page 42*]

Equation (18) provides an expression for $\dot{\theta}$ (or, equivalently, for the speed of the particle) in terms of the angle θ. As you have seen, this equation may be derived either by integrating the transverse component of the equation of motion or by using the law of conservation of mechanical energy, as in Exercise 10. We can now substitute this expression for $\dot{\theta}$ into the radial component of the equation of motion, namely Equation (15), to find the tension T in terms of θ. This gives

$$\begin{aligned} T &= mg\cos\theta + mr\dot{\theta}^2 \\ &= mg\cos\theta + \frac{mv_0^2}{r} - 2mg(1-\cos\theta) \end{aligned}$$

or $$T = \frac{mv_0^2}{r} + mg(3\cos\theta - 2). \tag{19}$$

Equations (18) and (19) have been derived on the assumption that the motion is always confined to a circle, so that the string is not slack. However, if the string does become slack during the motion then the analysis above is still valid for the initial part of the motion before the string becomes slack, that is, for the range of angles θ for which $T > 0$.

Exercise 11

This exercise refers to the model discussed in the text above, and requires the use of Equation (19).

(i) Show that if $|v_0| > \sqrt{5gr}$, then the string never goes slack.

(ii) If $v_0 = 2\sqrt{gr}$, find the angle θ for which the string goes slack.

Exercise 12

Consider a particle P of mass m, sliding on the outside of a fixed, smooth sphere of radius r. Assume that the particle has started from the highest point of the sphere with an initial horizontal velocity v_0, and that the radius from the centre of the sphere to the particle makes an angle θ with the upward vertical, as shown in Figure 10.

(i) Identify the forces acting on the particle when it is in contact with the sphere.

(ii) Use the law of conservation of mechanical energy to find an expression for $\dot{\theta}$ in terms of θ.

(iii) Use the radial component of the equation of motion to find, in terms of θ, the magnitude of the normal reaction from the sphere on the particle.

(iv) If $v_0 = \sqrt{\frac{1}{2}gr}$, show that the particle leaves the surface of the sphere when $\theta \simeq 33.6°$.

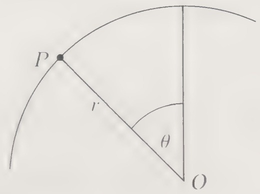


Figure 10

[*Solutions on page 42*]

Summary of Section 3

1. In terms of polar coordinates, the *radial* and *transverse* components of the position, velocity and acceleration of a particle moving in a circle, $r =$ constant, are as follows.

	Radial component	Transverse component
Position	r	0
Velocity	0	$r\dot{\theta}$
Acceleration	$-r\dot{\theta}^2$	$r\ddot{\theta}$

2. For circular motion with radius r and **angular velocity** $\omega = \dot{\theta}$, the velocity of the particle is $v = r\omega$, tangential to the circle. Both v and ω are positive for anticlockwise motion. The component of acceleration towards the centre of the circle is $r\omega^2 = v^2/r$. (For non-uniform motion, the acceleration also has a transverse component.)

3. For *uniform* circular motion, with radius r and constant angular velocity ω, the acceleration has magnitude $r\omega^2 = v^2/r$ and is directed towards the centre of the circle. The time taken for one complete revolution of the circle is $\tau = 2\pi/|\omega|$.

4 Using vector notation

This section brings together the topics of previous sections, with Subsection 4.1 following up Section 2 and the rest of Section 4 leading on from Section 3. As an aid to your study of subsequent units, we turn now to vector descriptions of some of the quantities introduced earlier. All that we shall do here is to re-express, in vector notation, some of the important definitions and equations encountered already, indicating where appropriate how these results may be generalized to three dimensions.

In previous sections the use of vector notation was played down. Instead, we described forces, velocities and accelerations in terms of their components in specified directions, such as 'down an inclined plane', or 'radially'. For moments and angular velocities, direction was indicated using the convention 'positive for anticlockwise, negative for clockwise', which makes sense only for two-dimensional situations. You have already seen, in *Unit 15*, how vectors may be used to describe force, velocity and acceleration, and in this section vector definitions will also be provided for moment (or *torque*) and *angular velocity.*

In order to use vector descriptions, it is convenient to relate these descriptions to a set of unit vectors in the directions of a right-handed set of axes. If we are concerned only with two-dimensional problems, then it is natural to choose these axes in such a way that the plane of the problem is the Cartesian (x, y)-plane. This means that the unit vector $\mathbf{k}$ is perpendicular to the plane of interest.

4.1 The torque of a force

In Section 2, for two-dimensional problems, we defined the *moment* of a force about a point as being equal to the product of the magnitude of the force and the perpendicular distance from the point to the line of action of the force. Further, the moment was identified as being in either the anticlockwise or clockwise rotational direction. As remarked earlier, the moment of a force is really a *vector* quantity, and in this subsection we shall derive a vector expression for the moment, using the vector cross product. Although this vector notation is not necessary in two dimensions, it is the vector form of the definition which is most easily generalized to three-dimensional problems.

The vector cross product was defined in *Unit 14* Subsection 3.5.

Consider a force $\mathbf{F}$, acting at a point which has position vector $\mathbf{r}$ with respect to the fixed origin O of a right-handed set of Cartesian axes, as shown in Figure 1. Let θ be the angle between the direction of $\mathbf{r}$ and the direction of $\mathbf{F}$. Then, from Figure 1, the force $\mathbf{F}$ will have an anticlockwise moment about O of magnitude

$$\Gamma^+ = F \times ON = Fr\sin\theta,$$

where F is the magnitude of $\mathbf{F}$. Using the definition of the vector cross product, this is equal to the magnitude of the cross product $\mathbf{r} \times \mathbf{F}$, that is,

$$\Gamma^+ = |\mathbf{r} \times \mathbf{F}|.$$

Being a vector, $\mathbf{r} \times \mathbf{F}$ also has a direction, which by the right-hand screw rule is out of the page, in the direction of the unit vector $\mathbf{k}$.

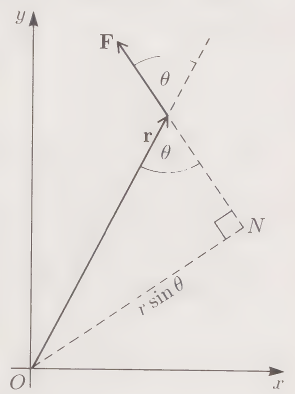


Figure 1

Exercise 1

For the force $\mathbf{F}$ shown in Figure 2, show that the magnitude of the clockwise moment of $\mathbf{F}$ about the point O is given by

$$\Gamma^- = |\mathbf{r} \times \mathbf{F}|.$$

What is the direction of the cross product $\mathbf{r} \times \mathbf{F}$ in this case?

[*Solution on page 42*]

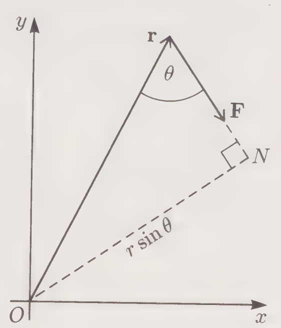


Figure 2

The above discussion and exercise suggest that the moment about a fixed point O of a force $\mathbf{F}$, acting at a point which has position vector $\mathbf{r}$ relative to O, may be represented by the cross product $\mathbf{r} \times \mathbf{F}$. This vector is perpendicular to the plane of $\mathbf{r}$ and $\mathbf{F}$. If the (x, y)-plane is chosen to contain $\mathbf{r}$ and $\mathbf{F}$, then an anticlockwise moment is represented by a vector pointing out of the plane (in the direction of the unit vector $\mathbf{k}$), whereas a clockwise moment is represented by one pointing into the plane (in the direction of $-\mathbf{k}$).

There is some confusion in the mathematical literature about the use of the word 'moment'. In some textbooks it describes both the vector quantity $\mathbf{r} \times \mathbf{F}$ and the magnitude of this vector. In this course we shall use 'moment' as we have done throughout Section 2 to mean the scalar magnitude, but always state whether it is anticlockwise or clockwise. The word *torque* is reserved for the vector quantity $\mathbf{\Gamma} = \mathbf{r} \times \mathbf{F}$ (see Figure 3).

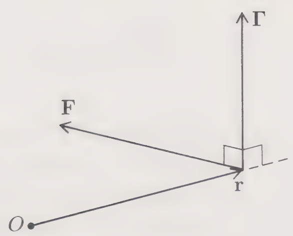


Figure 3

The torque of a force

The **torque** $\mathbf{\Gamma}$, about a fixed point O, of a force $\mathbf{F}$ acting at a point which has position vector $\mathbf{r}$ relative to the point O, is defined to be

$$\mathbf{\Gamma} = \mathbf{r} \times \mathbf{F}.$$

This definition applies even when the vectors $\mathbf{r}$ and $\mathbf{F}$ do not lie in the (x, y)-plane, as demonstrated by the first exercise below.

Exercise 2

The force $\mathbf{F} = 2\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$ acts at the point $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$. Find the torque about the origin of the force $\mathbf{F}$.

Exercise 3

Find the torque about the origin of the force $\mathbf{F} = 5\mathbf{i} - 3\mathbf{j}$, acting at the point $\mathbf{r} = \mathbf{i} + 2\mathbf{j}$. In the (x, y)-plane, is the moment anticlockwise or clockwise?

[*Solutions on page 42*]

You may recall that, for a rigid body in equilibrium under the action of a number of coplanar forces, the sum of all the anticlockwise moments is equal to the sum of all the clockwise moments. In other words,

$$\sum_i \Gamma_i^+ - \sum_j \Gamma_j^- = 0$$

This is Equation (2) of Section 2.

or
$$\sum_{\text{all forces}} (\Gamma_i^+ - \Gamma_j^-) = 0.$$

The sum on the left-hand side was referred to in Section 2 as the *total moment* of the forces acting on the body. The condition that the total moment is zero may be written in vector form as

$$\sum_{\text{all forces}} (\Gamma_i^+ - \Gamma_j^-)\mathbf{k} = \mathbf{0}$$

or
$$\sum_{\text{all forces}} \left(\Gamma_i^+\mathbf{k} + (-\Gamma_j^-\mathbf{k})\right) = \mathbf{0}.$$

Now $\Gamma_i^+\mathbf{k}$ is the torque of a force which has an anticlockwise moment, whereas $-\Gamma_j^-\mathbf{k}$ is the torque of a force with a clockwise moment. This means that the condition for equilibrium can be expressed simply as

$$\sum_i \mathbf{\Gamma}_i = \mathbf{0},$$

where the vector sum is taken over all the forces acting on the system, whether they have anticlockwise or clockwise moments. The left-hand side of this equation is the *total torque* acting on the body. For equilibrium, therefore, the total torque (or moment) about any point is equal to zero. You will see in *Unit 29* that this result is equally valid in three dimensions.

In Section 2 it was stated that the total moment of all the gravitational forces acting on a body could be evaluated by considering the weight of a representative particle concentrated at the centre of mass of the system. In the following exercise you are asked to prove this important result.

Exercise 4

Consider a system of n particles, where the ith particle has mass m_i and position vector $\mathbf{r}_i$ $(i = 1, 2, \ldots, n)$. Show that the total torque about the origin of the gravitational forces acting on the system is equal to

$$\mathbf{\Gamma}_{\text{grav}} = \mathbf{R} \times Mg\mathbf{k},$$

where $M = \sum_{i=1}^{n} m_i$ is the total mass of the system, $\mathbf{R}$ is the position of its centre of mass and $\mathbf{k}$ is a unit vector vertically downwards. (Recall, from the definition of centre of mass in *Unit 17* Subsection 2.3, that $M\mathbf{R} = \sum_{i=1}^{n} m_i\mathbf{r}_i$.)

[*Solution on page 43*]

To conclude this subsection, we shall illustrate how the vector notation can be used to solve static rigid-body problems. The procedure is summarized in the box below, and illustrated in Example 1. However, we recommend that for two-dimensional problems you should use the procedure developed in Subsection 2.3, using components and moments of the forces.

Solution of static rigid-body problems

1. Draw a diagram indicating all of the external forces acting on the body, and choose a suitable set of axes.
2. Express each external force in terms of the Cartesian unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.
3. Similarly, find the position vector of the point of action of each external force.
4. Equate to zero the total external force acting on the system.
5. Equate to zero the total torque acting on the system about any chosen point.

Example 1

A uniform ladder of mass M and length l stands on rough horizontal ground and rests against a smooth vertical wall. Find the minimum angle, θ, between the ladder and the horizontal for which the ladder can remain static, assuming that the coefficient of static friction between the base of the ladder and the ground is $\mu = 1$.

You considered this problem previously, as Exercise 6 of Section 2. You may like to compare the two solutions.

Solution

The four external forces acting on the ladder are (see Figure 4)

(i) the force of gravity $\mathbf{G}$, which may be considered to act at the centre of mass;

(ii) the normal reaction $\mathbf{R}$ from the ground, acting at the bottom of the ladder;

(iii) the normal reaction $\mathbf{S}$ from the wall, acting at the top of the ladder;

(iv) the frictional reaction $\mathbf{T}$, acting at the bottom of the ladder.

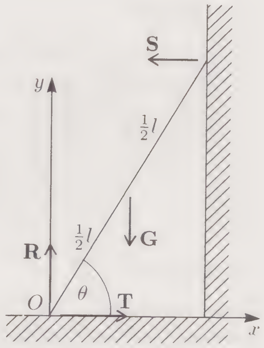


Figure 4

As two of the three unknown forces act at the bottom of the ladder, this is a convenient point about which to take torques, and so we choose the origin O to be at the bottom of the ladder. The x- and y-axes are chosen to be horizontal and vertical, as shown in Figure 4. With respect to these axes, we have

$$\mathbf{G} = -Mg\,\mathbf{j}, \quad \mathbf{R} = R\,\mathbf{j}, \quad \mathbf{S} = -S\,\mathbf{i}, \quad \mathbf{T} = T\,\mathbf{i},$$

where R, S and T are the respective magnitudes of the last three forces. The points of action of these forces have position vectors

$$\mathbf{r}_G = \tfrac{1}{2}l\cos\theta\,\mathbf{i} + \tfrac{1}{2}l\sin\theta\,\mathbf{j}, \quad \mathbf{r}_R = \mathbf{0}, \quad \mathbf{r}_S = l\cos\theta\,\mathbf{i} + l\sin\theta\,\mathbf{j}, \quad \mathbf{r}_T = \mathbf{0},$$

respectively. As the ladder is in equilibrium, the total external force acting on the ladder is zero, that is,

$$\mathbf{G} + \mathbf{R} + \mathbf{S} + \mathbf{T} = \mathbf{0} \quad \text{or} \quad -Mg\,\mathbf{j} + R\,\mathbf{j} - S\,\mathbf{i} + T\,\mathbf{i} = \mathbf{0},$$

from which it follows that

$$(T - S)\,\mathbf{i} + (R - Mg)\,\mathbf{j} = \mathbf{0}.$$

This vector equation gives the pair of scalar equations

$$S = T \tag{1}$$

and $$R = Mg. \tag{2}$$

Similarly, the total torque about the origin O is zero. Noting that the torques due to $\mathbf{R}$ and $\mathbf{T}$ about this point are zero, we obtain

$$\mathbf{r}_G \times \mathbf{G} + \mathbf{r}_S \times \mathbf{S} = \mathbf{0}$$

or $$(\tfrac{1}{2}l\cos\theta\,\mathbf{i} + \tfrac{1}{2}l\sin\theta\,\mathbf{j}) \times (-Mg\,\mathbf{j}) + (l\cos\theta\,\mathbf{i} + l\sin\theta\,\mathbf{j}) \times (-S\,\mathbf{i}) = \mathbf{0}.$$

Hence we find that

$$(-\tfrac{1}{2}Mgl\cos\theta + Sl\sin\theta)\,\mathbf{k} = \mathbf{0}$$

or $$S = \tfrac{1}{2}Mg\cot\theta.$$

From Equation (1), we then have

$$T = \tfrac{1}{2}Mg\cot\theta. \tag{3}$$

Now the law of friction is

$$T \le \mu R$$

which, on substituting the expressions for R and T from Equations (2) and (3), leads to the condition

$$\mu \ge \tfrac{1}{2}\cot\theta.$$

As the cotangent is a decreasing function in the interval $0 < \theta \le \frac{1}{2}\pi$, we conclude that

$$\theta \ge \operatorname{arccot} 2\mu = \operatorname{arccot} 2 \simeq 26.6^\circ. \quad \square$$

Exercise 5

A uniform beam OA of length l and mass m is free to turn in a vertical plane about a smooth hinge at O. It is supported in a horizontal static position by a light string, which is attached to end A of the beam and to a fixed point B vertically above O. If the inclination of the string to the horizontal is θ, use vector notation to find the reaction (magnitude and direction) at the hinge O and the tension in the string.

[*Solution on page 43*]

4.2 Motion in a circle (Television Subsection)

Study the following text, and attempt Exercises 6 and 7, before viewing the television programme.

In the television programme for this unit, the motion of a particle in a circle is considered using vector notation. As in Section 3, it is convenient here to use polar coordinates $[r, \theta]$ in the plane of the motion, but in order to do this in a vector context we need to introduce two vectors, $\mathbf{e}_r$ and $\mathbf{e}_\theta$, as shown in Figure 5. These are unit vectors in the radial (increasing r) and transverse (increasing θ) directions, respectively, where $[r, \theta]$ are the plane polar coordinates of the particle.

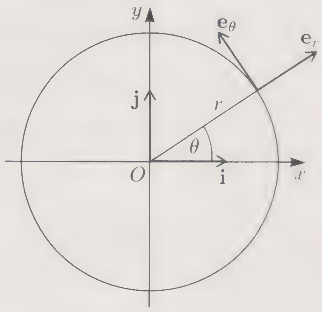


Figure 5

Exercise 6

Find expressions for the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ in terms of $\mathbf{i}$, $\mathbf{j}$ and θ. Further, find the cross product $\mathbf{e}_r \times \mathbf{e}_\theta$ in terms of $\mathbf{k}$.

[*Solution on page 43*]

A set of axes in the directions of the unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{k}$ is a right-handed set, but this set of axes moves with the particle. In this coordinate system, for example, the position vector of a particle moving in a circular path of radius r, with centre at the origin, is

$$\mathbf{r} = r\mathbf{e}_r.$$

In this expression the radius r is a constant, but the direction of the unit vector $\mathbf{e}_r$ varies as the circle is traversed, so that $\mathbf{e}_r$ is a function of time t.

Exercise 7

A particle moves in a circle of radius r, with centre at the origin. Use the expressions for the radial and transverse components of velocity and acceleration, as derived in Subsection 3.1, to write down the velocity $\mathbf{v} = \dot{\mathbf{r}}$ and acceleration $\mathbf{a} = \ddot{\mathbf{r}}$ of the particle in terms of r, the polar angle θ and the unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$.

[*Solution on page 43*]

The vector formulation of Newton's second law for a particle with mass m and position vector $\mathbf{r}$ is

$$\mathbf{F} = m\ddot{\mathbf{r}},$$

where $\mathbf{F}$ is the total force acting on the particle. This vector equation can be interpreted equally well in terms of Cartesian, polar or other coordinates. In *Unit 15* you gained familiarity with using the Cartesian form of Newton's second law, and in Section 3 of this unit you tackled problems using the radial and tangential components of the equation of motion. For planar motion, the total force acting on the particle may be expressed as

$$\mathbf{F} = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta,$$

where F_r and F_θ are the radial and transverse components of the force. Substituting into Newton's second law this expression for the force, and the expression for acceleration derived in Exercise 7, gives

$$F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta = -mr\dot{\theta}^2\mathbf{e}_r + mr\ddot{\theta}\mathbf{e}_\theta.$$

Now $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are perpendicular unit vectors, and so by taking the dot product of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ in turn with this equation, we obtain the radial and tangential components of Newton's second law as used in Section 3, namely

$$F_r = -mr\dot{\theta}^2 \quad \text{and} \quad F_\theta = mr\ddot{\theta}.$$

Now watch the television programme.
Read the following notes after viewing the programme.

TV28

Tom Smith began the programme with a reminder that a particle moving in a circle experiences an acceleration, even if its speed is constant. Mike Crampin then reviewed the basic theory of Subsection 3.1 for motion in a circle, using vector notation to describe the kinematics. In the remainder of the programme, two applications of the model were considered, namely, the motion of an artificial satellite in geostationary orbit, and a toy car or fairground vehicle 'looping the loop' on a circular track.

The vector notation for the kinematics of a particle in circular motion relies on use of the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$, which were introduced above and illustrated in Figure 5. The aim was to express the velocity and acceleration in terms of these vectors.

A particle moving anticlockwise around a circle of radius r, with polar angle θ, has travelled an arc distance $r\theta$ from its position at $\theta = 0$. Its speed is therefore $r\dot{\theta}$. Since its direction of travel is given by the unit vector $\mathbf{e}_\theta$, the velocity vector of the particle is

$$\mathbf{v} = \dot{\mathbf{r}} = r\dot{\theta}\mathbf{e}_\theta. \tag{4}$$

Another expression for $\mathbf{v}$ is obtained by direct differentiation of the position vector formula $\mathbf{r} = r\mathbf{e}_r$, which gives

Note that r is a constant, but that θ, $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are functions of time t.

$$\mathbf{v} = r\dot{\mathbf{e}}_r.$$

Comparison between this equation and Equation (4) shows that

$$\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta. \tag{5}$$

The next step is to find the corresponding formula for $\dot{\mathbf{e}}_\theta$. Now $\mathbf{e}_\theta$ may be thought of as being rigidly connected to $\mathbf{e}_r$, with both rotating at the same speed. Hence $|\dot{\mathbf{e}}_\theta| = |\dot{\mathbf{e}}_r| = |\dot{\theta}|$, from Equation (5). Also, $\mathbf{e}_\theta$ must change with time in such a way as to remain at 90° anticlockwise from $\mathbf{e}_r$. For $\dot{\theta} > 0$, the change in $\mathbf{e}_r$ is in the $\mathbf{e}_\theta$ direction, so $\mathbf{e}_\theta$ must then be changing in the $-\mathbf{e}_r$ direction, that is,

$$\dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e}_r. \tag{6}$$

The derivation of Equations (5) and (6) just given is that which was employed in the television programme. They may also be obtained by direct differentiation with respect

to time of the expressions for $\mathbf{e}_r, \mathbf{e}_\theta$ which you found in Exercise 6, namely

$$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}, \quad \mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}. \tag{7}$$

Equations (5)–(7) apply even when the planar motion is not confined to a circle. These formulas will be of use for the study of planetary motion in *Unit 30.*

By differentiating Equation (4), and using Equation (6), the acceleration vector may be written as

$$\begin{aligned}\mathbf{a} = \ddot{\mathbf{r}} &= r\ddot{\theta}\mathbf{e}_\theta + r\dot{\theta}\dot{\mathbf{e}}_\theta \\ &= -r\dot{\theta}^2\mathbf{e}_r + r\ddot{\theta}\mathbf{e}_\theta.\end{aligned} \tag{8}$$

As expected, Equations (4) and (8) are identical to the results obtained for the velocity and acceleration in Exercise 7.

As pointed out in the solution to Exercise 7, these results apply only for circular motion.

The first application of this model in the programme was to the problem of calculating how high above the Earth an artificial satellite has to be in order for it to remain in geostationary orbit, that is, fixed relative to a point on the Earth below. The force causing the circular motion in this case is the gravitational force of attraction on the satellite due to the presence of the Earth. It is a purely radial attractive force and has magnitude km/r^2, where k is a constant, m is the satellite's mass and r is its distance from the Earth's centre.

The gravitational force will be considered in more detail in *Unit 30.*

The value of the constant k can be evaluated by noting that the magnitude of the gravitational force is mg close to the Earth's surface, that is,

$$\frac{km}{R^2} = mg,$$

where $R \simeq 6.4 \times 10^6$ m is the radius of the Earth. It follows from this equation that

$$k = gR^2 \simeq 4.0 \times 10^{14}\,\mathrm{m}^3\,\mathrm{s}^{-2}.$$

The satellite is assumed to have a circular orbit, and the radial component of the equation of motion is

$$-\frac{km}{r^2} = -mr\dot{\theta}^2. \tag{9}$$

A geostationary satellite has an orbital period of 1 day, and so its angular speed is $\dot{\theta} = 2\pi/T$, where $T = 1$ day $\simeq 8.6 \times 10^4$ s. Substituting these values for k and $\dot{\theta}$ into Equation (9) gives $r \simeq 4.2 \times 10^7$ m, which means that the orbit is approximately 3.6×10^4 km above the Earth's surface.

The second application of the mathematics concerned a toy car moving on a curved track. The track contained an almost circular loop, which the car entered after being released at a certain height on a connecting sloping section of the track. If the car was released from sufficiently high up the sloping section then it 'looped the loop', staying in contact with the track throughout the circular portion. For lower initial heights, the car failed to complete the loop, falling off the track in the upper half of the circle. The aim was to calculate the minimum height for which the car manages to complete the loop.

During this motion the speed of the car varies, so that both components of Newton's second law are needed to solve the problem. Neglecting frictional forces, the radial and transverse components of the equation of motion are (see Figure 6)

The mathematics required to solve this problem is almost identical to that used in analysing the motion of a pendulum in Subsection 3.3.

$$-mr\dot{\theta}^2 = mg\cos\theta - R, \tag{10}$$

$$mr\ddot{\theta} = -mg\sin\theta, \tag{11}$$

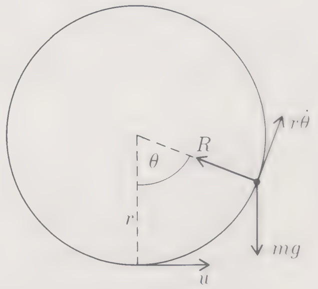


Figure 6

where R is the magnitude of the track's reaction on the car and m is the car's mass. Integrating Equation (11), and using the condition that the car has speed $r\dot{\theta} = u$ at the bottom of the track ($\theta = 0$), gives

$$\tfrac{1}{2}mr\dot{\theta}^2 = mg\cos\theta + \frac{mu^2}{2r} - mg.$$

Substituting for $\dot{\theta}$ in Equation (10), we obtain

$$R = 3mg\cos\theta - 2mg + \frac{mu^2}{r}. \tag{12}$$

The speed u of the car at the bottom of the track and the vertical height h above that point at which it is released are related, using conservation of total mechanical energy, by the formula

$$\tfrac{1}{2}mu^2 = mgh, \tag{13}$$

so that Equation (12) can be rewritten as

$$R = 3mg\cos\theta - 2mg + \frac{2mgh}{r}. \tag{14}$$

If the car 'loops the loop', then it is always in contact with the track, so that $R > 0$ for all values of θ in Equation (14). In particular, this applies at the top of the loop ($\theta = \pi$) where R has its minimum value. The corresponding inequality is equivalent to the condition

$$h > \tfrac{5}{2}r$$

for the car to 'loop the loop'. Equation (13) gives the corresponding condition for the speed u at the bottom of the track as $u > \sqrt{5gr}$.

Experiment shows that in practice a larger initial height than that predicted above is needed for the car to complete the loop. This discrepancy between prediction and reality is explained by the effects of friction, which were neglected in the model used in the programme.

The programme notes end here. The exercises below are follow-up work.

Consider further the mathematical model introduced in the programme for the motion of a toy car on a circular track, in which friction is neglected. Experience suggests that, for a low speed u at the bottom of the loop (corresponding to a low initial height h), the car will oscillate back and forth in the lower half of the circular track. For higher speeds u, the car will travel part of the way around the loop and then fall off, whereas for yet larger values of u, the car will 'loop the loop'. You saw in the television programme that the condition for this last type of behaviour is $h > \frac{5}{2}r$, which is equivalent to $u > \sqrt{5gr}$. We now ask you to derive the conditions for the other two types of behaviour.

From Equation (10), the magnitude of the reaction force from the track is

$$R = mg\cos\theta + mr\dot{\theta}^2.$$

The right-hand side of this equation is always positive for $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$, so the car cannot leave the track in the lower half of the circular loop. If the car is to leave the track then this must occur for $\theta > \frac{1}{2}\pi$.

Exercise 8

(i) Find the initial height h for which the toy car will just reach the horizontal level through the centre of the circular loop.

(ii) Find the corresponding value of the car's speed u at the bottom of the loop.

[*Solution on page 43*]

Combining the results of Exercise 8 with the previous results, we have found that the model predicts the following.

(i) For $h < r$ (or $u < \sqrt{2gr}$), the car oscillates back and forth in the lower half of the loop.

(ii) For $r < h < \frac{5}{2}r$ (or $\sqrt{2gr} < u < \sqrt{5gr}$), the car travels partly around the track and then falls off.

(iii) For $h > \frac{5}{2}r$ (or $u > \sqrt{5gr}$), the car 'loops the loop'.

Exercise 9 (The conical pendulum)

A *conical pendulum* is an apparatus in which a particle of mass m, attached by a light inextensible string of length l to a fixed point, moves around a horizontal circle below the fixed point. As the particle traverses the circle, the string traces out a cone whose semi-vertical angle α is the inclination of the string to the vertical (see Figure 7).

(i) Find the radius of the horizontal circle around which the particle moves.

(ii) Write down the radial component of the equation for the particle's circular motion.

(iii) Write down the vertical component of Newton's second law, remembering that the particle remains in a horizontal plane.

(iv) Use parts (ii) and (iii) to find the tension T in the string and the angular speed of the particle's circular motion.

[*Solution on page 43*]

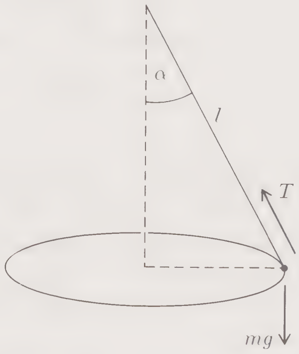


Figure 7

4.3 Angular velocity

In Subsection 3.1 we indicated that angular velocity, like linear velocity, is a vector quantity, but we did not then give a vector description for angular velocity. We shall complete this section by providing that vector description.

Exercise 10

A particle moves in a circle in the (x, y)-plane with centre at the origin. Show that the velocity $\dot{\mathbf{r}}$ of the particle satisfies the equation

$$\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r},$$

where the vector $\boldsymbol{\omega}$ is given by $\boldsymbol{\omega} = \dot{\theta}\mathbf{k}$.

[*Solution on page 44*]

The compact relation which you verified in Exercise 10 leads us to define the **angular velocity** $\boldsymbol{\omega}$ of a particle in circular motion centred on, and in a plane perpendicular to, the z-axis as

$$\boldsymbol{\omega} = \dot{\theta}\mathbf{k}.$$

The same definition may be applied to planar motion which is not circular.

The magnitude $\omega = |\boldsymbol{\omega}|$ of the angular velocity is known as the **angular speed** of the particle. Clearly, the angular speed is given by

$$\omega = |\dot{\theta}| = \begin{cases} \dot{\theta} & \text{if } \dot{\theta} \geq 0, \\ -\dot{\theta} & \text{if } \dot{\theta} < 0, \end{cases}$$

and the direction of the angular velocity vector is

$$\frac{\boldsymbol{\omega}}{\omega} = \begin{cases} \mathbf{k} & \text{if } \dot{\theta} > 0, \\ -\mathbf{k} & \text{if } \dot{\theta} < 0. \end{cases}$$

Thus the angular velocity is in the direction of $\mathbf{k}$ for anticlockwise motion, and $-\mathbf{k}$ for clockwise motion. This means that angular velocity is a vector whose direction is that of the axis of rotation, in the sense in which the rotation would drive a right-handed screw along the axis (see Figure 8). This is consistent with our convention for the torques of forces.

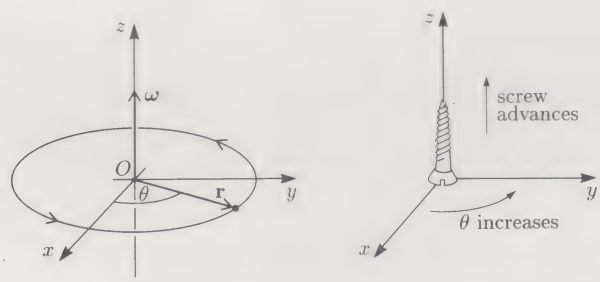


Figure 8

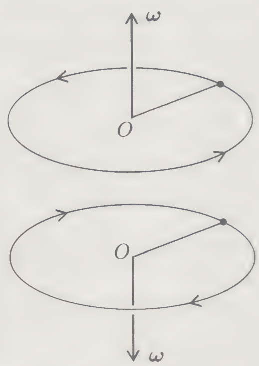


Figure 9

> The **angular velocity** $\boldsymbol{\omega}$ of a particle in circular motion centred on, and in a plane perpendicular to, the z-axis is $\boldsymbol{\omega} = \dot{\theta}\mathbf{k}$. This vector has magnitude equal to the angular speed $\omega = |\dot{\theta}|$, and direction along the axis of rotation, in the sense in which the rotation would drive a right-handed screw (see Figure 9).
>
> In terms of $\boldsymbol{\omega}$, the velocity of the particle is
>
> $$\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}. \tag{15}$$

Although the last formula was derived in Exercise 10 by taking the axis of rotation to be the z-axis and the motion to be in the (x, y)-plane, it applies wherever the origin O lies on the axis of rotation, as we shall now show. This result is useful in considering the rotation of an extended rigid body. The axis of such a rotation may be chosen as the z-axis, and the circular motion of each individual particle within the body then takes place in a plane parallel to the (x, y)-plane.

The rotation of rigid bodies is considered in *Unit 29*.

Consider such a particle motion in the plane $z = z_0$. In terms of the triad of unit vectors $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{k}$ (see Figure 10), the velocity and acceleration of the particle are always perpendicular to the $\mathbf{k}$-direction, and are therefore given as before by the equations

$$\mathbf{v} = \dot{\mathbf{r}} = r\dot{\theta}\mathbf{e}_\theta \quad \text{and} \quad \mathbf{a} = \ddot{\mathbf{r}} = -r\dot{\theta}^2\mathbf{e}_r + r\ddot{\theta}\mathbf{e}_\theta.$$

These equations may be verified by differentiating the expression below for the position vector $\mathbf{r}$.

However, the position of the particle is now a height z_0 above the point in the (x, y)-plane with position vector $r\mathbf{e}_r$, so the position vector of the particle is

$$\mathbf{r} = r\mathbf{e}_r + z_0\mathbf{k}.$$

Thus with $\boldsymbol{\omega} = \dot{\theta}\mathbf{k}$, we find that

$$\begin{aligned}\boldsymbol{\omega} \times \mathbf{r} &= \dot{\theta}\mathbf{k} \times (r\mathbf{e}_r + z_0\mathbf{k}) \\ &= r\dot{\theta}\mathbf{k} \times \mathbf{e}_r + \dot{\theta}z_0\mathbf{k} \times \mathbf{k} \\ &= r\dot{\theta}\mathbf{e}_\theta = \dot{\mathbf{r}},\end{aligned}$$

as before.

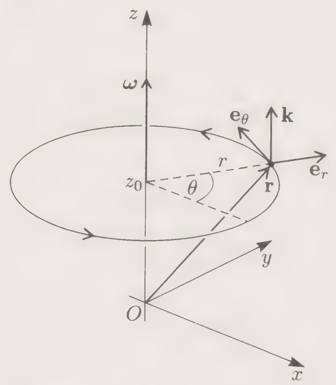


Figure 10

Exercise 11

A particle moves at a constant rate around a circle which is centred on, and whose plane is perpendicular to, the z-axis. Its angular speed is $|\dot{\theta}| = 2\,\mathrm{rad\,s^{-1}}$ and, to an observer who looks along the $\mathbf{k}$-direction, the particle appears to move in an anticlockwise sense.

(i) Find the angular velocity vector of the particle.

(ii) If the position vector of the particle at a certain instant is $3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$, find its velocity vector at that instant both in terms of the unit vectors $\mathbf{i}$, $\mathbf{j}$ and in terms of the unit vector $\mathbf{e}_\theta$.

[*Solution on page 44*]

Summary of Section 4

1. The **torque** $\boldsymbol{\Gamma}$, about a fixed point O, of a force $\mathbf{F}$ acting at a point which has position vector $\mathbf{r}$ relative to the point O, is defined to be

$$\boldsymbol{\Gamma} = \mathbf{r} \times \mathbf{F}.$$

2. The torque of all the gravitational forces acting on a system of particles or extended body can be evaluated by considering the total weight of the system to be concentrated at its centre of mass.

3. If a system of particles or extended body is in equilibrium, then the total external force is equal to zero, as is the total torque of the external forces about any fixed point.

4. With respect to plane polar coordinates $[r, \theta]$, the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ (in the radial and transverse directions, respectively) are given by

$$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}, \quad \mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}.$$

The vectors $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{k}$ form a right-handed triad of unit vectors. The directions of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ vary with time, so that

$$\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta \quad \text{and} \quad \dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e}_r.$$

5. The position, velocity and acceleration of a particle moving in a circle of constant radius r in the (x, y)-plane, with centre at the origin, are

$$\begin{aligned}&\mathbf{r} = r\mathbf{e}_r, \\ &\mathbf{v} = \dot{\mathbf{r}} = r\dot{\theta}\mathbf{e}_\theta \\ \text{and} \quad &\mathbf{a} = \ddot{\mathbf{r}} = -r\dot{\theta}^2\mathbf{e}_r + r\ddot{\theta}\mathbf{e}_\theta.\end{aligned}$$

6. The **angular velocity** $\boldsymbol{\omega}$ for a particle in circular motion is a vector whose magnitude is equal to the angular speed, and whose direction is along the axis of rotation, in the sense in which the rotation would drive a right-handed screw. If the unit vector $\mathbf{k}$ is perpendicular to the plane of motion and the centre of the circle lies on the z-axis, then

$$\boldsymbol{\omega} = \dot{\theta}\mathbf{k}.$$

7. For circular motion with angular velocity $\boldsymbol{\omega}$, the velocity of a particle is given by

$$\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r},$$

where $\mathbf{r}$ is the position vector of the particle relative to an origin on the axis of rotation.

5 End of unit exercises

Section 2

Exercise 1

A light horizontal rod, of length 1.2 m, is supported by two vertical props, each 0.3 m from an end of the rod. What masses, suspended from the ends of the rod, will produce reactions of magnitudes $3.2g$ N and $0.8g$ N from the props?

Exercise 2

A uniform sphere of radius a is kept at rest on a smooth plane, which is inclined to the horizontal at an angle α, by means of a string attached to a point on the surface of the sphere and to a point higher up the plane.

(i) By taking moments about the centre of the sphere, show that the line of action of the force provided by the string passes through the centre of the sphere.

(ii) The tension in the string is to be no greater than the magnitude of the weight of the sphere. By taking moments about the point of contact of the sphere and the plane, show that the length of the string must be at least $a(\sec\alpha - 1)$.

Exercise 3

A uniform beam AB, of length $2l$ and weight W, rests tangentially against the rim of a smooth fixed circular disc of radius a (where $a < 2l/\sqrt{3}$), whose plane is the vertical plane through the beam. The lower end, A, of the beam is in contact with a horizontal plane passing through the lowest point O of the disc. The end A of the beam is acted upon by a frictional force of magnitude F (directed towards the point O), and the inclination of the beam to the horizontal is 60°. By considering the forces acting on the beam, show that $F = Wl/(4a)$.

Exercise 4

A uniform ladder of mass M and length l stands on horizontal ground and rests against a vertical wall. The coefficient of static friction between the ladder and the wall is μ_W, whilst that between the ladder and the ground is μ_G. Show that, for the ladder to remain static, the angle θ between the ladder and the ground must satisfy the inequality

$$2\tan\theta \geq \frac{1}{\mu_G} - \mu_W.$$

Exercise 5

A uniform, smooth ladder rests with its extremities against a vertical wall and the horizontal ground. It is held fixed by a rope, one end of which is attached to a rung of the ladder one quarter of the way up, the other end being fixed to the wall at ground level. The base and top of the ladder are at distances a and b respectively from the base of the wall. If the magnitudes of the reactions between the ladder and the ground and wall are denoted respectively by P and Q, show that

$$\frac{Q}{P} = \frac{3a}{5b}.$$

Exercise 6

A static circular cylinder of weight W, with its axis horizontal, is supported in contact with a rough vertical wall by a string wrapped partly around it and attached to a point on the wall. The string makes an angle α with the wall. Show that the coefficient of static friction between the wall and the cylinder must be at least $\operatorname{cosec}\alpha$, and that the magnitude of the normal reaction between the cylinder and the wall is $W\tan\frac{1}{2}\alpha$.

Exercise 7

A tailor's dummy of mass 20 kg is at rest on a 10° slope, facing sideways across the slope as shown in Figure 1. Its feet are 0.7 m apart, while its centre of mass is 1.2 m from the sloping ground and mid-way between the feet. Find the magnitudes of the normal reaction forces exerted by the ground on the dummy's feet.

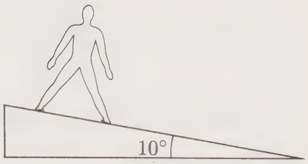


Figure 1

Exercise 8

A light ladder 3.9 m long stands on a horizontal floor and rests against a smooth vertical wall. The bottom of the ladder is 1.5 m from the wall, and the coefficient of static friction between the ladder and the floor is 0.25. The end rungs are each 0.3 m from an end of the ladder. What is the minimum mass of a person standing on the bottom rung which prevents the ladder from slipping when a person of mass 100 kg stands on the top rung?

[*Solutions on page 44*]

Section 3

Exercise 9

Helicopter blades rotate in such a way that the constant speed of the tips is 200 m s^{-1}. This is the case for all helicopter blades, regardless of their length. Calculate the corresponding number of revolutions per second for each of the following:

(i) a Chinook blade, of length 9.14 m;

(ii) a Sikorsky blade, of length 8.45 m;

(iii) a Westland Lynx blade, of length 6.4 m.

Exercise 10

A long-playing gramophone record is designed to spin at a constant angular speed of $33\frac{1}{3}$ revolutions per minute (rpm). Relative to the record, the stylus travels in a spiral, but in a simple model it may be considered to travel in a circle on each revolution. Show that the stylus travels faster relative to the record when near the edge of the disc (of radius 15 cm) than it does when near the centre (at a radius of 6 cm), by calculating its tangential speed at each of these radii.

Exercise 11

In contrast with the gramophone record of Exercise 10, a digital compact disc (CD) player varies the rotational speed of the disc, with the signal tracker moving relative to the disc at a constant speed of 1.25 m s^{-1}. By considering the signal tracker to be travelling relative to the disc in circles (rather than spirally), show that the rotational speed of a compact disc, with respective maximum and minimum diameters 12 cm and 4 cm, varies between about 200 rpm and 600 rpm.

Exercise 12

A particle is attached by a light inextensible string of length 1 m to a fixed point on a smooth horizontal table, so that it moves around a circle of radius 1 m in a horizontal plane. The string can support only a tension equal to fifteen times the weight of the particle. Show that the greatest possible number of revolutions per second is just under two.

Exercise 13

Three particles, each of mass 1 kg, are at the vertices of an equilateral triangle whose sides are taut, inextensible strings of length 0.2 m. The system is on a smooth, horizontal table, and rotates uniformly about its centre at a rate of 5 revolutions per second. Find the tension in each of the strings.

Exercise 14

A smooth hemispherical bowl of radius r, whose lowest point is at A, is fixed with its rim uppermost and horizontal. A particle of mass m is projected along the inner surface of the bowl towards A, so that its subsequent motion is in a vertical plane through A. The particle is projected from a point at a vertical height $\frac{1}{2}r$ above A, with initial speed $\sqrt{gr}$.

(i) Show that the particle will just reach the top of the bowl.

(ii) Find the magnitude of the reaction of the bowl on the particle when the particle is at a vertical height $\frac{1}{3}r$ above A.

[*Solutions on page 46*]

Appendix: Solutions to the exercises

Solutions to the exercises in Section 1

1.

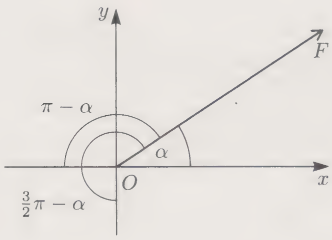


Figure 1

The angle between the force and the negative x-axis is $\pi - \alpha$, so the component of the force in this direction is

$$F\cos(\pi - \alpha) = F(\cos\pi\cos\alpha + \sin\pi\sin\alpha) = -F\cos\alpha.$$

The angle between the force and the negative y-axis is $\frac{3}{2}\pi - \alpha$, so the component of the force in this direction is

$$F\cos(\tfrac{3}{2}\pi - \alpha) = -F\sin\alpha.$$

2.

Figure 2

The component of the force in the direction East is $F\cos\frac{1}{4}\pi = F/\sqrt{2}$. The component of the force in the direction North-West is 0, as this direction is perpendicular to that of the force.

3. Using the notation of Figure 7 of Section 1, from $\triangle PAC$ we have $\widehat{APC} = \frac{1}{2}\pi - \theta$ and hence $\widehat{BPC} = \frac{1}{2}\pi - \widehat{APC} = \theta$. So the components of the forces in the required directions are as given in the table below.

Force	Normal to plane	Down plane
Force of gravity	$-mg\cos\theta$	$mg\sin\theta$
Normal reaction	N	0
Force of friction	0	$-S$

4.

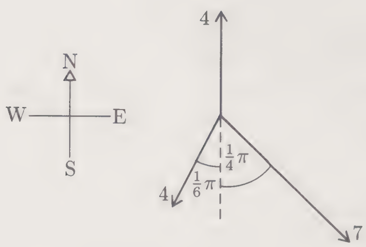


Figure 3

(i) Denote the components of the fourth force in the easterly and northerly directions by X and Y respectively. As the particle is in equilibrium, the sum of the components of the four forces in any direction must be zero. Resolving the forces due East gives

$$X + 7\sin\tfrac{1}{4}\pi - 4\sin\tfrac{1}{6}\pi = 0 \quad \text{or} \quad X \simeq -2.95.$$

Resolving the forces due North, we have

$$Y + 4 - 7\cos\tfrac{1}{4}\pi - 4\cos\tfrac{1}{6}\pi = 0 \quad \text{or} \quad Y \simeq 4.41.$$

(ii)

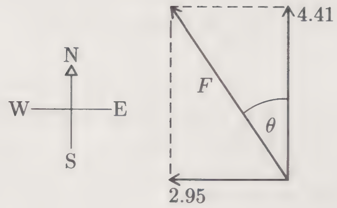


Figure 4

The components of the fourth force are 4.41 N due North and -2.95 N due East (or 2.95 N due West). So the magnitude of the fourth force is

$$F = \sqrt{(4.41)^2 + (2.95)^2} \simeq 5.31\,\text{N},$$

and its direction is North θ West, where

$$\theta = \arctan(2.95/4.41) \simeq 33.75°.$$

5.

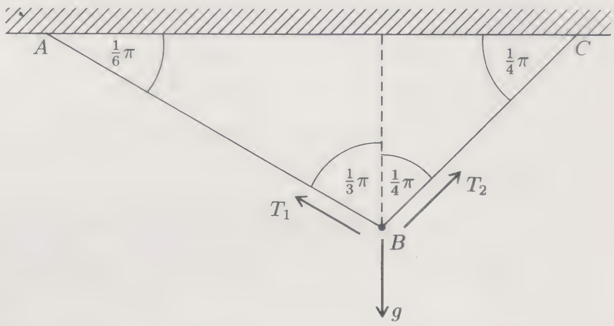


Figure 5

The forces acting on the particle are shown in the figure above, where the tensions in the two strings are denoted by T_1 and T_2 respectively. Resolving the forces in the direction perpendicular to string BC, we have

$$T_1\sin\tfrac{5}{12}\pi - g\sin\tfrac{1}{4}\pi = 0,$$

from which we obtain

$$T_1 = \frac{g\sin\frac{1}{4}\pi}{\sin\frac{5}{12}\pi} \simeq 7.18\,\text{N}.$$

Resolving the forces in the direction perpendicular to string AB gives

$$T_2\sin\tfrac{5}{12}\pi - g\sin\tfrac{1}{3}\pi = 0.$$

Hence

$$T_2 = \frac{g\sin\frac{1}{3}\pi}{\sin\frac{5}{12}\pi} \simeq 8.80\,\text{N}.$$

6.

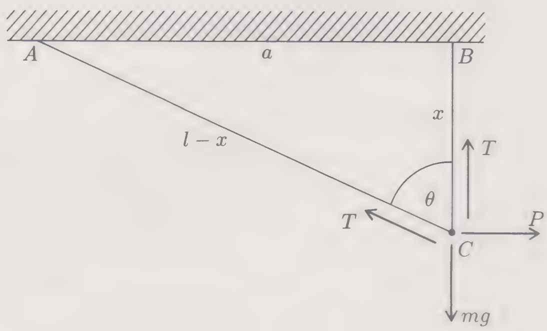


Figure 6

Denote the tension in the string by T, and let the ring C be at a distance x below B. The forces acting on the ring are as shown in the figure above.

Applying Pythagoras' Theorem to $\triangle ABC$ gives

$$a^2 + x^2 = (l - x)^2.$$

It follows that

$$x = \frac{l^2 - a^2}{2l} \quad \text{and} \quad l - x = \frac{l^2 + a^2}{2l}.$$

So if $\widehat{ACB} = \theta$, then

$$\cos\theta = \frac{x}{l - x} = \frac{l^2 - a^2}{l^2 + a^2},$$

$$\sin\theta = \frac{a}{l - x} = \frac{2al}{l^2 + a^2}.$$

Resolving the forces vertically, we obtain

$$T + T\cos\theta - mg = 0,$$

from which

$$T = \frac{mg}{1 + \cos\theta} = \frac{mg(l^2 + a^2)}{2l^2}.$$

Resolving the forces horizontally, we have

$$P - T\sin\theta = 0.$$

Hence

$$P = T\sin\theta = \frac{mga}{l}.$$

Solutions to the exercises in Section 2

1. If Jack's weight is W N then Jill's weight is $\frac{5}{6}W$ N. For horizontal balance to occur, Jill must sit on the opposite half of the see-saw from Jack, at a distance $\frac{6}{5}l$ from the pivot. (It is assumed that the see-saw is sufficiently long for this to occur.)

2.

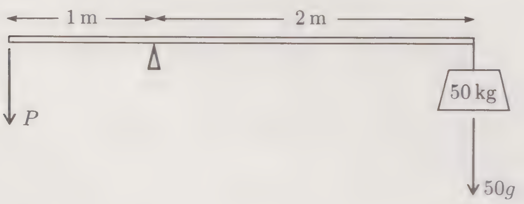


Figure 1

The beam is described as 'light', so that its mass may be neglected. Let the magnitude of the required force be P N, as shown in the figure above. (You may have drawn the figure the other way round.)

The anticlockwise moment Γ^+ about the pivot is caused by the force of magnitude P, whose line of action is at a perpendicular distance 1 from the pivot. So

$$\Gamma^+ = P \times 1 = P.$$

Similarly, the clockwise moment Γ^- about the pivot is

$$\Gamma^- = 50g \times 2 = 100g.$$

For the beam to balance, we must have

$$\Gamma^+ - \Gamma^- = 0,$$

which leads directly to

$$P = 100g.$$

Thus the required force acting vertically downwards at the end of the shorter arm has magnitude $100g$ N.

3. If the plank does not overturn then it will be in contact with the quay for 3.5 m of its length, and there will be reactive forces between the quay and the plank along this length. However, if the plank does tip over then it will do so about the edge A of the quay, and the only reactive force will be at this pivotal point. We consider the limiting situation, with the plank still horizontal but about to start tipping, when the only reactive force between the quay and the plank is at A. Let the magnitude of the required weight permitting a person to reach the outer end of the plank be W N.

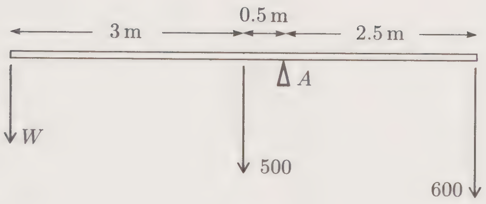


Figure 2

Considering anticlockwise moments about A, we have

$$\Gamma_1^+ = 3.5W \quad \text{and} \quad \Gamma_2^+ = 0.5 \times 500 = 250.$$

The clockwise moment about A is

$$\Gamma^- = 600 \times 2.5 = 1500.$$

So for the plank to be horizontal we must have

$$3.5W + 250 - 1500 = 0.$$

Therefore

$$7W = 2500 \quad \text{or} \quad W \simeq 357.1.$$

Therefore a weight of at least 357.2 N must be placed at the plank's landward end (and the plank must be sufficiently strong) if the plank is not to tip (or break).

4. With the notation of Figure 6 of Section 2, we again note that it must be the tension in the left-hand string which is zero, that is, $T_1 = 0$.

Resolving the forces vertically, we have

$$T_2 = 40g + W.$$

Taking moments about the centre of mass, we obtain

$$T_2 \times 1 - W \times 1.5 = 0 \quad \text{or} \quad T_2 = 1.5W.$$

Eliminating T_2 between these two equations gives

$$1.5W = 40g + W \quad \text{or} \quad W = 80g,$$

as before.

5.

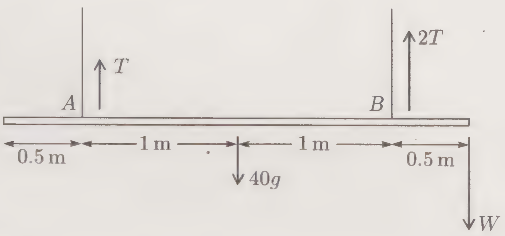


Figure 3

The notation is as defined in the figure above, which indicates all of the external forces acting on the beam. By considering the turning effects of the forces about the centre of mass, we can see that weight must be applied at the end of the beam opposite to the half on which the string with the smaller tension is attached. (The opposite assumption leads to a contradiction.)

Taking moments about the right-hand end of the beam (which ensures that only one unknown quantity enters the equation), we have

$$40g \times 1.5 - (T \times 2.5 + 2T \times 0.5) = 0,$$

which leads to

$$T = \tfrac{120}{7}g.$$

Resolving the forces vertically, we obtain

$$T + 2T - 40g - W = 0,$$

so that

$$\begin{aligned} W &= 3T - 40g \\ &= \tfrac{360}{7}g - 40g \\ &= \tfrac{80}{7}g \simeq 112. \end{aligned}$$

Hence the magnitude of the weight that must be placed at one end of the beam is approximately 112 N.

6.

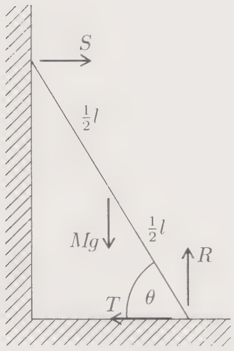


Figure 4

As in Example 3 of Section 2, we obtain

$$R = Mg \quad \text{and} \quad T = \tfrac{1}{2}Mg\cot\theta.$$

Now the law of friction (*Unit 15* Section 3) is $T \le \mu R$, so that

$$\mu \ge \tfrac{1}{2}\cot\theta.$$

As the cotangent is a decreasing function in the interval $0 < \theta \le \frac{1}{2}\pi$, we have

$$\theta \ge \operatorname{arccot} 2\mu = \operatorname{arccot} 2 \simeq 26.6°.$$

Hence the minimum angle with the horizontal for which the ladder can remain static is about 26.6°.

7.

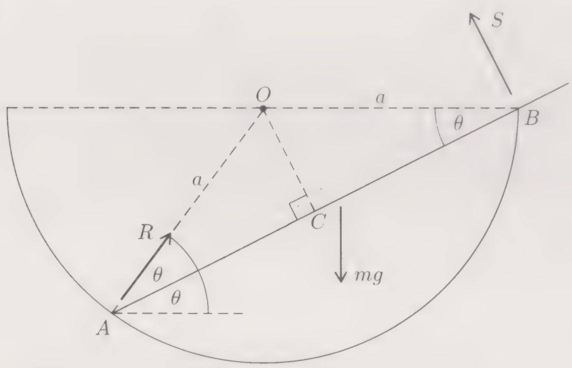


Figure 5

With the notation as in the figure above, we have $\widehat{OBA} = \theta$. Also $\triangle OAB$ is isosceles (since $OA = OB = a$), and so

$$BC = a\cos\theta \quad \text{and} \quad AB = 2BC = 2a\cos\theta.$$

The bowl is smooth, and so the reaction at A is normal to the bowl and the reaction at B is normal to the rod. The magnitudes of these reactions are denoted by R and S respectively.

Resolving the forces along the rod (in order to eliminate one of the unknown reactions) we have

$$R\cos\theta - mg\sin\theta = 0 \quad \text{or} \quad R = mg\tan\theta.$$

Resolving horizontally gives

$$R\cos 2\theta - S\sin\theta = 0,$$

so that

$$S = \frac{R\cos 2\theta}{\sin\theta} = \frac{mg\cos 2\theta}{\cos\theta}.$$

Finally, taking moments about the point A, we obtain

$$S \times 2a\cos\theta - mg \times a\cos\theta = 0 \quad \text{or} \quad S = \tfrac{1}{2}mg.$$

Equating the two expressions for S yields the required condition

$$2\cos 2\theta = \cos\theta.$$

Now $\cos 2\theta = 2\cos^2\theta - 1$, so that $\cos\theta$ satisfies the quadratic equation

$$4\cos^2\theta - \cos\theta - 2 = 0.$$

The solutions are given by

$$\cos\theta = \frac{1 \pm \sqrt{1 + 4 \times 4 \times 2}}{8} = \tfrac{1}{8}(1 \pm \sqrt{33}).$$

As $0 < \theta < \frac{1}{2}\pi$, we have

$$\theta = \arccos\left(\tfrac{1}{8}(1 + \sqrt{33})\right) \simeq 32.5°.$$

8.

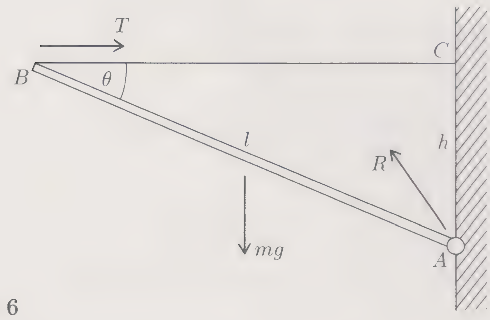


Figure 6

The forces acting on the beam are as shown in the figure above, where the beam makes an angle θ with the horizontal. We again take moments about the point A, in order to eliminate the unknown reaction of magnitude R. This gives

$$mg \times \tfrac{1}{2}l\cos\theta - T \times h = 0.$$

Hence we have

$$T = \frac{mgl\cos\theta}{2h} = \frac{mg\sqrt{l^2 - h^2}}{2h}.$$

Solutions to the exercises in Section 3

1. Using the chain rule, we have

$$\frac{d}{dt}(\cos\theta(t)) = \frac{d}{d\theta}(\cos\theta)\frac{d\theta}{dt} = -\dot\theta\sin\theta$$

and

$$\frac{d}{dt}(\sin\theta(t)) = \frac{d}{d\theta}(\sin\theta)\frac{d\theta}{dt} = \dot\theta\cos\theta.$$

On differentiating

$$\mathbf{r} = r\cos\theta(t)\mathbf{i} + r\sin\theta(t)\mathbf{j},$$

we obtain

$$\dot{\mathbf{r}} = r\frac{d}{dt}(\cos\theta(t))\mathbf{i} + r\frac{d}{dt}(\sin\theta(t))\mathbf{j}$$
$$= -r\dot\theta\sin\theta\,\mathbf{i} + r\dot\theta\cos\theta\,\mathbf{j}.$$

2. **(i)** The radial component of the vector $\dot x\,\mathbf{i}$ is $\dot x\cos\theta$, while its transverse component is $-\dot x\sin\theta$.

(ii) The radial component of the vector $\dot y\,\mathbf{j}$ is $\dot y\sin\theta$, while its transverse component is $\dot y\cos\theta$.

(iii) The component of the velocity vector $\dot{\mathbf{r}} = \dot x\,\mathbf{i} + \dot y\,\mathbf{j}$ in any direction is equal to the sum of the components of $\dot x\,\mathbf{i}$ and $\dot y\,\mathbf{j}$ in that direction. So the radial component of the velocity $\dot{\mathbf{r}}$ is

$$\dot x\cos\theta + \dot y\sin\theta,$$

and its transverse component is

$$-\dot x\sin\theta + \dot y\cos\theta.$$

3. The particle is moving on the circle $r = 1$. Also $\theta = t^2$, so that

$$\dot\theta = 2t \quad \text{and} \quad \ddot\theta = 2.$$

Hence, from Table 1 in Section 3, the velocity of the particle has radial component 0 and transverse component

$$r\dot\theta = 2t,$$

while its acceleration has radial component

$$-r\dot\theta^2 = -4t^2$$

and transverse component

$$r\ddot\theta = 2.$$

4. **(i)** The transverse component of the particle's velocity is $v = r\dot{\theta}$. As the radial component is zero, the particle's speed (which is the magnitude of the velocity vector) is equal to the magnitude of the transverse component, which is $|v|$.

(ii) In terms of $v = r\dot{\theta}$, the radial component of the particle's acceleration is

$$-r\dot{\theta}^2 = -v^2/r,$$

as required.

5. **(i)** We have $\theta = \omega t$ with ω constant, so that

$$\dot{\theta} = \omega \quad \text{and} \quad \ddot{\theta} = 0.$$

Hence, from Table 1 in Section 3, the velocity has radial component 0 and transverse component $r\omega$. Also, the acceleration has radial component $-r\omega^2$ and transverse component 0.

Note that the particle's velocity is tangential to the circle and has magnitude $r|\omega|$, while its acceleration is directed towards the centre of the circle and has magnitude $r\omega^2$.

(ii) From the solution to Exercise 4(i), the particle's speed is

$$|v| = |r\dot{\theta}| = r|\omega|,$$

which is a constant. The total distance travelled in one complete revolution is $2\pi r$, and so the time taken for each revolution is

$$\tau = \frac{2\pi r}{r|\omega|} = \frac{2\pi}{|\omega|}.$$

6. The minute hand of the clock makes one complete revolution (clockwise!) in 1 hour = 3600 s. So the fly's angular velocity is

$$\omega = -\frac{2\pi}{3600} \simeq -1.75 \times 10^{-3}\,\mathrm{rad\,s^{-1}},$$

where the minus sign indicates that the motion is clockwise. The fly's velocity is

$$r\omega = -\frac{\pi}{3600} \simeq -8.73 \times 10^{-4}\,\mathrm{m\,s^{-1}},$$

where the minus sign again indicates motion in the clockwise sense around the circle (the velocity is always tangential to the circular path of the particle). The acceleration has magnitude

$$r\omega^2 = 0.5\left(\frac{\pi}{1800}\right)^2 \simeq 1.52 \times 10^{-6}\,\mathrm{m\,s^{-2}},$$

and is directed towards the centre of the clock.

7.

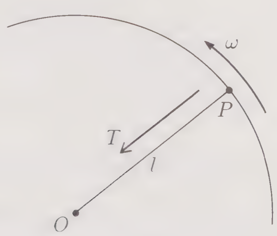


Figure 1

The problem is one of two-dimensional motion in the plane of the circle, as shown in the figure above. Let T be the tension in the string. As the particle of mass m is in uniform circular motion, with radius l and angular velocity ω, the radial component of acceleration is $-l\omega^2$. So the radial component of Newton's second law gives

$$-T = -ml\omega^2 \quad \text{or} \quad T = ml\omega^2.$$

8.

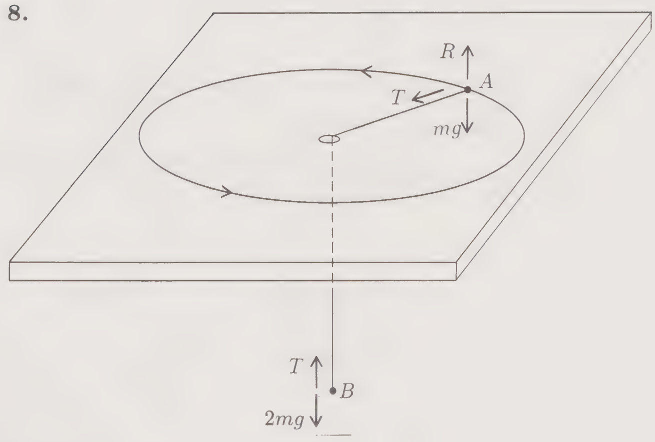


Figure 2

Let the particles be labelled A and B, as shown in the figure above, and let T be the tension in the string. First consider Particle B. As it is static, we have

$$T = 2mg.$$

Consider now the uniform circular motion of Particle A. The radial component of Newton's second law gives

$$-T = -mv^2/r.$$

Eliminating T between these two equations, we obtain

$$v^2 = 2gr.$$

It follows that the constant speed of Particle A is

$$|v| = \sqrt{2gr} = \sqrt{2 \times 9.81 \times 0.4} \simeq 2.80\,\mathrm{m\,s^{-1}}.$$

9.

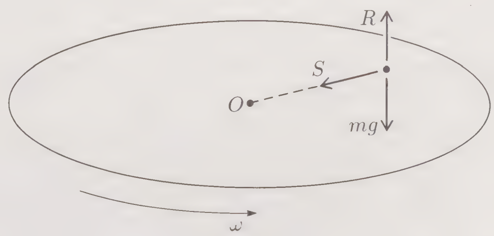


Figure 3

If the disc rotates through two complete revolutions per second, then its angular velocity is

$$\omega = 2 \times 2\pi = 4\pi\,\mathrm{rad\,s^{-1}}.$$

The forces acting on the particle, as shown on the diagram above, are:

(i) the force of gravity, of magnitude mg vertically downwards;

(ii) the normal reaction from the disc, of magnitude R vertically upwards;

(iii) the frictional force, of magnitude S and with direction in the horizontal plane of the disc.

As the particle does not move vertically, we have

$$R = mg.$$

Since the particle stays at the same point on the disc, at a distance r from the centre, say, it performs uniform circular motion with angular velocity $\omega = 4\pi$. Hence its acceleration has magnitude $r\omega^2$, and is directed towards the centre of the disc. The only force with a horizontal component is the frictional force. So this frictional force is directed towards the centre of the disc, and has magnitude

$$S = mr\omega^2.$$

Now the law of friction is

$$S \le \mu R$$

which, from the expressions above for R and S, leads to

$$r \le \frac{\mu g}{\omega^2} = \frac{g}{32\pi^2} \simeq 0.031\,\text{m}.$$

Thus the maximum possible distance of the object from the centre of the disc is about 3.1 cm.

10.

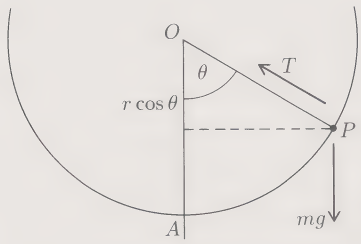


Figure 4

In circular motion, the velocity of the particle is $v = r\dot{\theta}$, so the kinetic energy of the particle is

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}mr^2\dot{\theta}^2.$$

If we use the lowest point A of the particle's motion as the datum, then the gravitational potential energy is

$$U = mg \times \text{height of particle above } A$$
$$= mg(r - r\cos\theta).$$

So the total mechanical energy of the particle is

$$\tfrac{1}{2}mr^2\dot{\theta}^2 + mgr(1-\cos\theta) = E.$$

Initially, at the point A, the particle has velocity v_0, which means that $\dot{\theta} = v_0/r$ when $\theta = 0$. Hence we have

$$E = \tfrac{1}{2}mv_0^2 + mgr(1-1) = \tfrac{1}{2}mv_0^2.$$

Thus conservation of mechanical energy together with the initial condition gives

$$\tfrac{1}{2}mr^2\dot{\theta}^2 + mgr(1-\cos\theta) = \tfrac{1}{2}mv_0^2$$

or

$$\dot{\theta}^2 = \frac{v_0^2}{r^2} - \frac{2g}{r}(1-\cos\theta),$$

which is Equation (18) of Section 3.

11. Equation (19) of Section 3 is

$$T = \frac{mv_0^2}{r} + mg(3\cos\theta - 2).$$

(i) For the string never to go slack, we must have $T > 0$ for all values of θ. The minimum value of T occurs when $\cos\theta$ has its minimum value of -1 at $\theta = \pm\pi$. This minimum value is

$$T_{\min} = \frac{mv_0^2}{r} - 5mg.$$

It follows that the string never goes slack provided that

$$\frac{mv_0^2}{r} - 5mg > 0 \quad \text{or} \quad |v_0| > \sqrt{5gr}.$$

(ii) If $v_0 = 2\sqrt{gr}$, then the tension in the string during the circular motion is

$$T = 4mg + mg(3\cos\theta - 2)$$
$$= mg(3\cos\theta + 2).$$

The string goes slack when $T = 0$, that is, when

$$\theta = \arccos(-\tfrac{2}{3}) \simeq 132°.$$

Here the positive solution for θ has been chosen (rather than $\theta = -\arccos(-\frac{2}{3}) \simeq -132°$) because the given initial velocity v_0 is positive, so that the particle is initially moving in the anticlockwise sense around the circle.

12. (i)

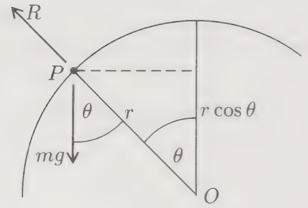


Figure 5

The forces acting on the particle are the force of gravity, of magnitude mg vertically downwards, and the normal reaction from the sphere, of magnitude R. These are shown in the diagram above.

(ii) The kinetic energy of the particle is

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}mr^2\dot{\theta}^2.$$

Using the centre O of the sphere as the datum, the gravitational potential energy is

$$U = mgr\cos\theta.$$

The normal reaction does not affect the conservation of mechanical energy, as it is always perpendicular to the direction of the particle's motion. Therefore the total mechanical energy is

$$\tfrac{1}{2}mr^2\dot{\theta}^2 + mgr\cos\theta = E.$$

Initially, we have $\theta = 0$ and $\dot{\theta} = v_0/r$. This initial condition gives

$$E = \tfrac{1}{2}mv_0^2 + mgr.$$

Hence we obtain

$$\tfrac{1}{2}mr^2\dot{\theta}^2 + mgr\cos\theta = \tfrac{1}{2}mv_0^2 + mgr$$

or $$\dot{\theta}^2 = \frac{v_0^2}{r^2} + \frac{2g}{r}(1-\cos\theta).$$

(iii) The radial component of the equation of motion is

$$R - mg\cos\theta = -mr\dot{\theta}^2.$$

Hence the normal reaction has magnitude

$$R = mg\cos\theta - mr\dot{\theta}^2$$
$$= mg\cos\theta - \frac{mv_0^2}{r} - 2mg(1-\cos\theta)$$
$$= mg(3\cos\theta - 2) - \frac{mv_0^2}{r}.$$

(iv) If $v_0 = \sqrt{\frac{1}{2}gr}$, then

$$R = mg(3\cos\theta - 2) - \tfrac{1}{2}mg$$
$$= mg(3\cos\theta - \tfrac{5}{2}).$$

The particle will leave the surface when $R = 0$, that is, when

$$\theta = \arccos(\tfrac{5}{6}) \simeq 33.6°.$$

Solutions to the exercises in Section 4

1. Using the notation of Figure 2 of Section 4, and putting $F = |\mathbf{F}|$, the clockwise moment of $\mathbf{F}$ about the point O is

$$\Gamma^- = F \times ON = Fr\sin\theta = |\mathbf{r} \times \mathbf{F}|.$$

According to the right-hand screw rule, the direction of $\mathbf{r} \times \mathbf{F}$ is into the page, in the direction of the unit vector $-\mathbf{k}$.

2. By definition, the torque is

$$\mathbf{\Gamma} = \mathbf{r} \times \mathbf{F} = (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \times (2\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}).$$

Using the method of *Unit 14* Subsection 3.7 for the cross product of vectors given in component form, this becomes

$$\mathbf{\Gamma} = (6 + 12)\mathbf{i} + (6 + 3)\mathbf{j} + (-4 + 4)\mathbf{k} = 18\mathbf{i} + 9\mathbf{j}.$$

3. Proceeding as in the previous exercise, the torque is

$$\mathbf{\Gamma} = \mathbf{r} \times \mathbf{F}$$
$$= (\mathbf{i} + 2\mathbf{j}) \times (5\mathbf{i} - 3\mathbf{j})$$
$$= (-3 - 10)\mathbf{k} = -13\mathbf{k}.$$

As the torque is in the direction of $-\mathbf{k}$, the moment is *clockwise* in the (x, y)-plane.

4. The gravitational force acting on the ith particle is $\mathbf{F}_i = m_i g\mathbf{k}$, whose torque about the origin is

$$\mathbf{\Gamma}_i = \mathbf{r}_i \times \mathbf{F}_i = \mathbf{r}_i \times m_i g\mathbf{k}.$$

The total torque acting on the system about the origin is therefore

$$\mathbf{\Gamma}_{\text{grav}} = \sum_{i=1}^{n} \mathbf{\Gamma}_i = \sum_{i=1}^{n} (\mathbf{r}_i \times m_i g\mathbf{k}) = g\left(\sum_{i=1}^{n} m_i\mathbf{r}_i\right) \times \mathbf{k}.$$

From the definition of centre of mass, we have

$$\sum_{i=1}^{n} m_i\mathbf{r}_i = M\mathbf{R},$$

so that

$$\mathbf{\Gamma}_{\text{grav}} = gM\mathbf{R} \times \mathbf{k} = \mathbf{R} \times Mg\mathbf{k},$$

as required. (Note that this result is valid only if the gravitational field is uniform.)

5.

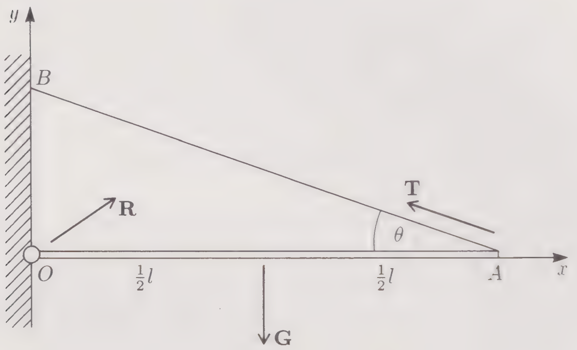


Figure 1

The external forces are as shown in the figure above. We choose O as the origin, with the x- and y-axes horizontal and vertical. With respect to these axes, the forces are

(i) the force of gravity, $\mathbf{G} = -mg\mathbf{j}$, acting at the point $\mathbf{r}_G = \frac{1}{2}l\,\mathbf{i}$;

(ii) the force caused by the tension T in the string, $\mathbf{T} = -T\cos\theta\,\mathbf{i} + T\sin\theta\,\mathbf{j}$, acting at the point $\mathbf{r}_T = l\,\mathbf{i}$;

(iii) the reaction $\mathbf{R}$ at the hinge, of unknown magnitude and direction, acting at the point $\mathbf{r}_R = \mathbf{0}$.

As the system is in equilibrium, the total torque about the origin is zero. So we obtain

$$\tfrac{1}{2}l\,\mathbf{i} \times (-mg\mathbf{j}) + l\,\mathbf{i} \times (-T\cos\theta\,\mathbf{i} + T\sin\theta\,\mathbf{j}) = \mathbf{0}$$

or $\quad (-\frac{1}{2}mgl + Tl\sin\theta)\mathbf{k} = \mathbf{0}$.

Hence

$$T = \frac{mg}{2\sin\theta} = \tfrac{1}{2}mg\operatorname{cosec}\theta.$$

Also, the total external force on the beam is zero, which gives

$$-\,mg\,\mathbf{j} + (-T\cos\theta\,\mathbf{i} + T\sin\theta\,\mathbf{j}) + \mathbf{R} = \mathbf{0}.$$

From this we find that

$$\begin{aligned}\mathbf{R} &= T\cos\theta\,\mathbf{i} + (mg - T\sin\theta)\mathbf{j}\\ &= \frac{mg\cos\theta}{2\sin\theta}\mathbf{i} + (mg - \tfrac{1}{2}mg)\mathbf{j}\\ &= \tfrac{1}{2}mg\cot\theta\,\mathbf{i} + \tfrac{1}{2}mg\,\mathbf{j}.\end{aligned}$$

The magnitude of this reaction is

$$R = \tfrac{1}{2}mg\sqrt{\cot^2\theta + 1} = \tfrac{1}{2}mg\operatorname{cosec}\theta,$$

and the angle between its direction and the horizontal is

$$\arctan\left(\frac{\frac{1}{2}mg}{\frac{1}{2}mg\cot\theta}\right) = \arctan(\tan\theta) = \theta.$$

6.

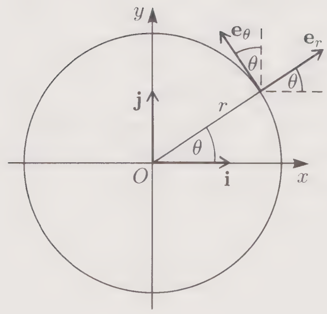


Figure 2

Note that $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are unit vectors, and therefore have magnitude 1. By finding their components in the $\mathbf{i}$- and $\mathbf{j}$-directions, we obtain

$$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} \quad \text{and} \quad \mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}.$$

We can use these expressions to evaluate the cross product $\mathbf{e}_r \times \mathbf{e}_\theta$. However, it is easier to note that $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are two perpendicular unit vectors, so that $|\mathbf{e}_r \times \mathbf{e}_\theta| = 1$. By the right-hand screw rule, the direction of $\mathbf{e}_r \times \mathbf{e}_\theta$ is out of the page, and so

$$\mathbf{e}_r \times \mathbf{e}_\theta = \mathbf{k}.$$

7. From Table 1 in Section 3, the radial and transverse components of velocity are 0 and $r\dot\theta$ respectively. So the velocity vector is

$$\mathbf{v} = \dot{\mathbf{r}} = 0\mathbf{e}_r + r\dot\theta\mathbf{e}_\theta = r\dot\theta\mathbf{e}_\theta.$$

Similarly, the acceleration vector is

$$\mathbf{a} = \ddot{\mathbf{r}} = -r\dot\theta^2\mathbf{e}_r + r\ddot\theta\mathbf{e}_\theta.$$

(Note that these expressions hold only for *circular* motion, where r is constant. More general expressions for planar motion will be derived in *Unit 30*.)

8. **(i)** The condition for the car just to reach the horizontal level through the centre of the loop is $\dot\theta = 0$ at $\theta = \frac{1}{2}\pi$. So, from Equations (10) and (14) of Section 4, we find that

$$0 = -2mg + \frac{2mgh}{r} \quad \text{or} \quad h = r.$$

(ii) From the conservation of mechanical energy (Equation (13) of Section 4), we have

$$\tfrac{1}{2}mu^2 = mgh.$$

Corresponding to $h = r$, we therefore obtain

$$u = \sqrt{2gr}.$$

9. **(i)** The radius of the particle's circular path is

$$r = l\sin\alpha.$$

(ii) The two forces acting on the particle are gravity and the force due to the tension T in the string. Only the second of these has a (horizontal) radial component, whose magnitude is $T\sin\alpha$. So the radial equation of motion is

$$-\,mr\dot\theta^2 = -T\sin\alpha.$$

As $r = l\sin\alpha$ from part (i), this reduces to

$$ml\dot\theta^2 = T.$$

(iii) The particle has no vertical component of acceleration, and so the total force acting on the particle must have a zero vertical component. Hence we obtain

$$T\cos\alpha - mg = 0.$$

(iv) From part (iii), the tension is given by

$$T = mg\sec\alpha.$$

Substituting this into the result of part (ii), the angular speed is

$$|\dot\theta| = \sqrt{\frac{g\sec\alpha}{l}} = \sqrt{\frac{g}{l\cos\alpha}}.$$

10. In circular motion, the position vector of the particle is given by $\mathbf{r} = r\mathbf{e}_r$. Hence we have

$$\boldsymbol{\omega} \times \mathbf{r} = \dot{\theta}\mathbf{k} \times r\mathbf{e}_r = r\dot{\theta}\mathbf{k} \times \mathbf{e}_r = r\dot{\theta}\mathbf{e}_\theta,$$

since $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{k}$ form a right-handed triad of unit vectors. But from Equation (4) of Section 4, the velocity of the particle is $\dot{\mathbf{r}} = r\dot{\theta}\mathbf{e}_\theta$. It follows that

$$\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r},$$

as required.

11. (i)

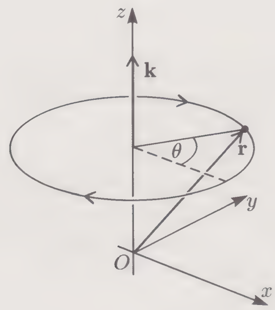


Figure 3

From the figure above, θ is decreasing during the motion, so that $\dot{\theta} = -2$ and the particle's angular velocity vector is

$$\boldsymbol{\omega} = \dot{\theta}\mathbf{k} = -2\mathbf{k}.$$

(ii) From Equation (15) of Section 4, the particle's velocity vector can be calculated as

$$\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r} = -2\mathbf{k} \times (3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) = 8\mathbf{i} - 6\mathbf{j}.$$

Alternatively, since the radius of the circular path is $\sqrt{3^2 + 4^2} = 5$, the velocity vector is given by

$$\dot{\mathbf{r}} = r\dot{\theta}\mathbf{e}_\theta = 5 \times (-2)\mathbf{e}_\theta = -10\mathbf{e}_\theta.$$

Solutions to the exercises in Section 5

1.

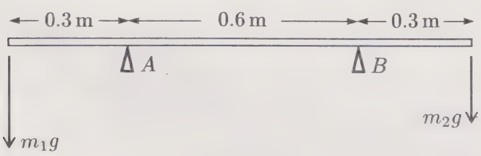


Figure 1

Let the unknown masses be m_1 and m_2, with respective weights m_1g and m_2g as shown in the figure above. Suppose that the props A and B respectively provide the upward reaction forces of magnitudes $3.2g$ and $0.8g$.

Then resolving vertically downwards gives

$$m_1g + m_2g - 3.2g - 0.8g = 0$$

or $\quad m_1 + m_2 = 4. \qquad (1)$

Taking moments about A, we have

$$0.3 \times m_1g + 0.6 \times 0.8g - 0.9 \times m_2g = 0$$

or $\quad 3m_1 + 4.8 - 9m_2 = 0.$

Substituting for m_1 from Equation (1) produces

$$3(4 - m_2) + 4.8 - 9m_2 = 0 \quad \text{or} \quad m_2 = 1.4,$$

from which $m_1 = 2.6$. So the required masses are 2.6 kg and 1.4 kg respectively.

2. (i) The lines of action of the gravitational force due to the sphere's mass and the normal reaction of the plane on the sphere both pass through the centre O of the sphere, so that each has zero moment about this point. For equilibrium, the string force must also have zero moment about O. Since this force has non-zero magnitude, its line of action passes through O.

(ii)

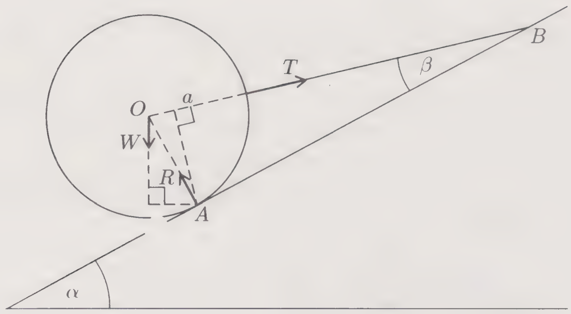


Figure 2

Let A be the point of contact between the sphere and the plane, and B be the point at which the string is attached to the plane, where β is the angle between the string and the plane. Let W be the weight of the sphere, T the tension in the string and R the magnitude of the normal reaction on the sphere at A.

Taking moments about A gives

$$Wa\sin\alpha - Ta\cos\beta = 0. \qquad (2)$$

Considering $\triangle OBA$, we see that

$$OB = \frac{a}{\sin\beta} = a\,\mathrm{cosec}\,\beta,$$

so that the length of the string is

$$OB - a = a(\mathrm{cosec}\,\beta - 1).$$

From Equation (2) and the given condition $T \leq W$, we have

$$\cos\beta \geq \sin\alpha,$$

from which the minimum value of $\cos\beta$ is

$$\sin\alpha = \cos(\tfrac{1}{2}\pi - \alpha).$$

It follows that the maximum value of $\sin\beta$ is

$$\sin(\tfrac{1}{2}\pi - \alpha) = \cos\alpha,$$

and hence that the minimum value of $\mathrm{cosec}\,\beta$ is $\sec\alpha$. The minimum possible value for the string length $a(\mathrm{cosec}\,\beta - 1)$ is therefore

$$a(\sec\alpha - 1),$$

as required.

3.

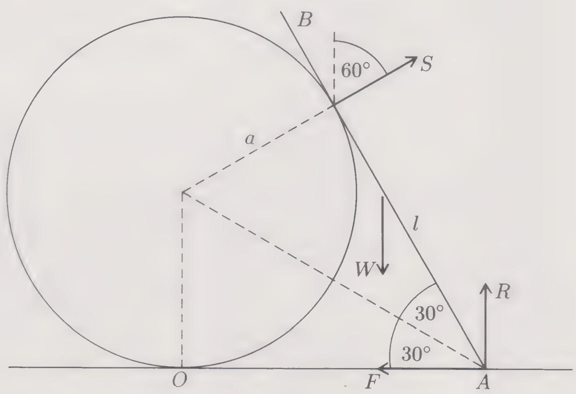


Figure 3

Let R and S respectively be the magnitudes of the normal reactions of the ground and the disc on the beam. The other forces acting on the beam are its weight, of magnitude W, and the frictional force of magnitude F. Resolving forces horizontally gives

$$S\sin 60^\circ - F = 0.$$

Taking moments about A, we obtain

$$Wl\cos 60^\circ - S\frac{a}{\tan 30^\circ} = 0.$$

On substituting for S from above, this produces

$$\tfrac{1}{2}Wl = 2Fa,$$

so that $F = Wl/(4a)$ as required. (Notice that there is no need to resolve forces vertically in this solution.)

4.

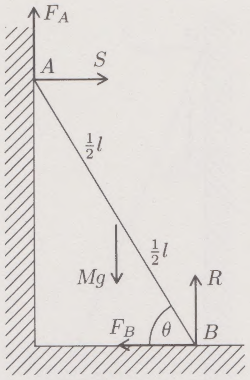


Figure 4

The forces acting on the ladder AB are shown in the figure above. Since the ladder is uniform, the gravitational force due to its mass acts vertically downwards through its mid-point. There are normal reactions of magnitudes S and R, and frictional forces of magnitudes F_A and F_B acting at A and B respectively, as shown in the figure. The ladder is static, so we may follow the procedure of resolving forces both vertically and horizontally, and taking moments.

Resolving horizontally gives

$$S - F_B = 0. \tag{3}$$

On resolving vertically, we have

$$F_A + R - Mg = 0. \tag{4}$$

Taking moments about B, we obtain

$$Mg \times \tfrac{1}{2}l\cos\theta - Sl\sin\theta - F_A l\cos\theta = 0$$

or $\quad \tan\theta = \dfrac{Mg - 2F_A}{2S}.$

Substituting for Mg from Equation (4) produces

$$\tan\theta = \frac{R - F_A}{2S}. \tag{5}$$

However, if the ladder is to remain static then, by the law of friction, we must have

$$F_A \le \mu_W S \quad \text{and} \quad F_B \le \mu_G R,$$

so that, using Equation (3),

$$R - F_A \ge \frac{F_B}{\mu_G} - \mu_W S = \left(\frac{1}{\mu_G} - \mu_W\right) S.$$

Equation (5) then gives

$$2\tan\theta \ge \frac{1}{\mu_G} - \mu_W,$$

as required.

5.

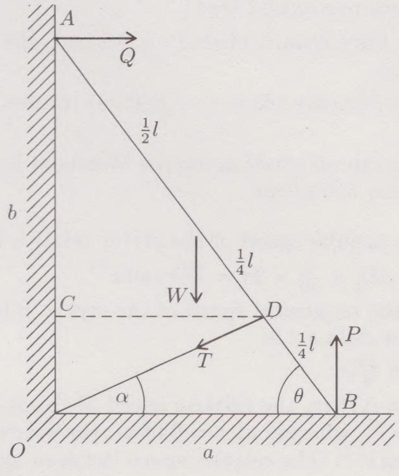


Figure 5

Let W be the weight of the ladder and let T be the tension in the rope. Suppose that the ladder AB has length l, and that it is inclined at an angle θ to the horizontal, while the rope makes an angle α with the horizontal, as shown in the figure.

Resolving forces vertically, we have

$$P - W - T\sin\alpha = 0. \tag{6}$$

Resolving forces horizontally gives

$$Q - T\cos\alpha = 0. \tag{7}$$

Taking moments about the point D where the rope is attached to the ladder, we obtain

$$W \times \tfrac{1}{4}l\cos\theta + P \times \tfrac{1}{4}l\cos\theta - Q \times \tfrac{3}{4}l\sin\theta = 0$$

or $\quad W + P = 3Q\tan\theta. \tag{8}$

From Equations (6) and (7), it follows that

$$\begin{aligned}\tan\alpha &= \frac{P - W}{Q}\\ &= \frac{P}{Q} - \frac{3Q\tan\theta - P}{Q} \quad \text{(from Equation (8))}\\ &= \frac{2P}{Q} - 3\tan\theta. \end{aligned} \tag{9}$$

Now $\tan\theta = b/a$, while $\tan\alpha$ may be expressed in terms of a and b as follows.

Let $OC = x$ and $CD = y$. Then by similar triangles ($\triangle ABO$ and $\triangle ADC$), we have

$$\frac{b - x}{b} = \frac{y}{a} = \frac{\frac{3}{4}l}{l} = \frac{3}{4},$$

leading to

$$y = \tfrac{3}{4}a, \quad b - x = \tfrac{3}{4}b \quad \text{and} \quad x = \tfrac{1}{4}b.$$

Then

$$\tan\alpha = \frac{x}{y} = \frac{b}{3a},$$

so that, using Equation (9),

$$\frac{b}{3a} = \frac{2P}{Q} - \frac{3b}{a} \quad \text{or} \quad \frac{10b}{3a} = \frac{2P}{Q}.$$

Thus $Q/P = 3a/(5b)$, as required.

6.

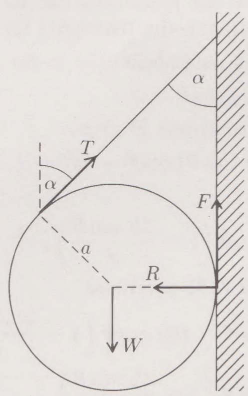


Figure 6

Let T be the tension in the string, R the magnitude of the normal reaction at the wall and F the magnitude of the frictional force which prevents the cylinder from slipping down the wall. Let a be the radius of the cylinder.

Consider the forces acting on the cylinder. Resolving vertically, we have

$$T\cos\alpha + F - W = 0, \tag{10}$$

while resolving horizontally gives

$$T\sin\alpha - R = 0. \tag{11}$$

Taking moments about the centre of the cylinder, we obtain

$$aF - aT = 0 \quad \text{or} \quad T = F.$$

Substituting for T in Equations (10) and (11) produces

$$F\cos\alpha + F = W \quad \text{and} \quad F\sin\alpha = R. \tag{12}$$

For the system to remain motionless, we require $F \le \mu R$, where μ is the coefficient of static friction between the wall and the cylinder. Hence, using the second of Equations (12),

$$\mu \ge \frac{F}{R} = \frac{F}{F\sin\alpha} = \operatorname{cosec}\alpha,$$

showing that the coefficient of static friction must be at least $\operatorname{cosec}\alpha$.

Also, from Equations (12), we see that

$$R = F\sin\alpha = \frac{W\sin\alpha}{1+\cos\alpha} = \frac{2W\sin\frac{1}{2}\alpha\cos\frac{1}{2}\alpha}{2\cos^2\frac{1}{2}\alpha} = W\tan\tfrac{1}{2}\alpha,$$

so the magnitude of the normal reaction is $W\tan\frac{1}{2}\alpha$.

7.

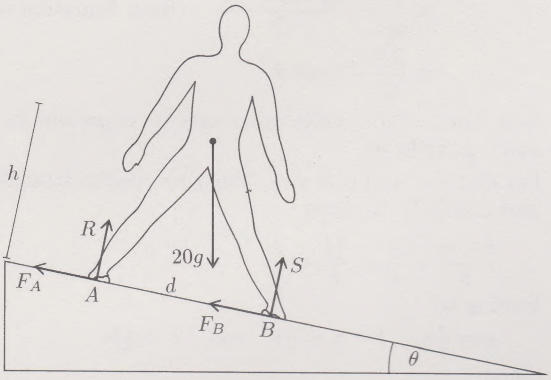


Figure 7

Let A and B respectively be the points of contact of the upper and lower feet of the dummy with the sloping plane. Let h be the distance of the centre of mass from the plane (1.2 m), d the distance AB (0.7 m) and θ the angle of slope (10°). The normal reaction forces and frictional forces at A and B are as shown in the figure. By resolving forces perpendicular to the plane and taking moments about B, it is possible to eliminate the frictional forces from consideration.

Resolving forces perpendicular to the sloping plane, we have

$$R + S - 20g\cos\theta = 0. \tag{13}$$

Taking moments about B gives

$$20g(\tfrac{1}{2}d - h\tan\theta)\cos\theta - dR = 0,$$

leading to

$$R = 10g\cos\theta\left(1 - \frac{2h\tan\theta}{d}\right). \tag{14}$$

Using Equation (13) produces

$$S = 20g\cos\theta - 10g\cos\theta\left(1 - \frac{2h\tan\theta}{d}\right) = 10g\cos\theta\left(1 + \frac{2h\tan\theta}{d}\right). \tag{15}$$

Substituting the given values for h, d and θ into Equations (14) and (15), we find that the normal reactions at A, B have respective magnitudes 38.2 N and 155.0 N, so that the normal reaction on the lower foot is roughly four times that on the upper foot. (Note, from Equation (14), that if $\tan\theta = d/(2h)$ then the upper foot experiences no normal reaction. For this angle of slope the dummy is on the point of toppling over, since its centre of mass is vertically above the lower foot.)

8.

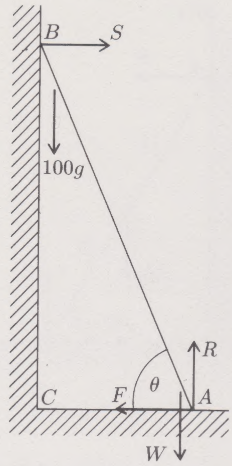


Figure 8

Since the ladder is light, its mass may be neglected. The normal reactions of the wall and floor on the ladder have magnitudes S and R respectively, as shown in the figure. There is a horizontal frictional force of magnitude F acting at the base A of the ladder. Let W be the weight of the person standing on the bottom rung of the ladder.

Since $AB = 3.9$ m and $AC = 1.5$ m, it follows that $\triangle ABC$ is a 5:12:13 triangle. Hence $BC = 3.6$ m, and

$$\cos\theta = \tfrac{5}{13}, \quad \sin\theta = \tfrac{12}{13}.$$

Resolving forces vertically, we have

$$R = W + 100g, \tag{16}$$

while resolving forces horizontally and applying the law of friction gives

$$S = F \le 0.25R.$$

Taking moments about A, we obtain

$$100g \times (3.9 - 0.3) \times \tfrac{5}{13} + W \times 0.3 \times \tfrac{5}{13} - S \times 3.6 = 0$$

or $$12S = 100g \times 12 \times \tfrac{5}{13} + \tfrac{5}{13}W. \tag{17}$$

From Equation (16), we have

$$W = R - 100g \ge 4S - 100g = \frac{1}{3}\left(\frac{6000g}{13} + \frac{5W}{13}\right) - 100g,$$

using Equation (17). Thus $\frac{34}{3}W \ge 700g$, so that the minimum weight of the person on the bottom rung is $61.76g$ N. The corresponding mass is about 62 kg.

9. In each case, the angular frequency of rotation is $\omega = 200/r\ \text{rad s}^{-1}$, where r is the length of the blade. This corresponds to a rotation rate of $\omega/(2\pi) = 100/(\pi r)$ revolutions per second (rps).

(i) For the Chinook blade ($r = 9.14$ m), the rate of rotation is 3.48 rps.

(ii) The Sikorsky blade ($r = 8.45$ m) rotates at a rate of 3.77 rps.

(iii) The rate of rotation for the Westland Lynx blade ($r = 6.4$ m) is 4.97 rps.

10. The angular speed of the stylus relative to the record is

$$\omega = 33\tfrac{1}{3} \times \tfrac{1}{60} \times 2\pi = \tfrac{10}{9}\pi\ \text{rad s}^{-1},$$

so that the tangential speed of the stylus, relative to the record, at radius r is

$$\omega r = \tfrac{10}{9}\pi r.$$

With $r = 0.15$ m, the relative speed of the stylus is $0.524\ \text{m s}^{-1}$, whereas at $r = 0.06$ m, the corresponding speed is $0.209\ \text{m s}^{-1}$. The relative speed between stylus and record therefore decreases from $52.4\ \text{cm s}^{-1}$ at the edge of the disc to $20.9\ \text{cm s}^{-1}$ near the centre.

11. Relative to the compact disc, the angular speed of the tracking head is $\omega = 1.25/r\,\mathrm{rad\,s^{-1}}$, where r m is the radius. This corresponds to a rotational rate of

$$\frac{1.25}{2\pi r} \times 60 = \frac{37.5}{\pi r}\ \mathrm{rpm},$$

giving 198.9 rpm for $r = 0.06$ m (12 cm diameter) and 596.8 rpm for $r = 0.02$ m (4 cm diameter). So the compact disc player varies its rotational speed between 200 rpm and 600 rpm approximately.

12. As in Exercise 7 of Section 3, the tension in the string is given by

$$T = mr\omega^2,$$

where here $r = 1$ m. From the given condition $T \le 15mg$, we obtain

$$\omega^2 \le 15g.$$

If n is the maximum possible number of revolutions per second then the corresponding value for the angular speed is $\omega = 2\pi n$, so that

$$n^2 = \frac{15g}{4\pi^2} \simeq 3.73, \qquad \text{or} \qquad n \simeq 1.93.$$

Thus the greatest possible number of revolutions per second is just under two.

13.

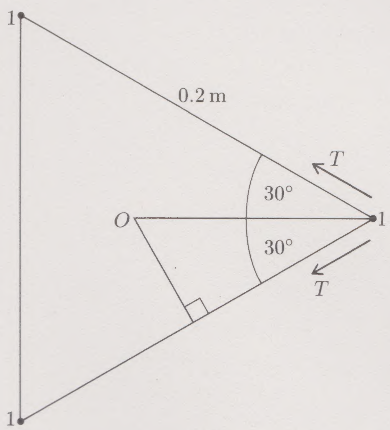


Figure 9

By symmetry, the tension in each string is the same. Let T denote this tension, and consider any one of the particles. The total component of force radially inwards is

$$2T\cos 30^\circ = T\sqrt{3}.$$

The particle travels on a circle of radius

$$r = \frac{0.1}{\cos 30^\circ} = \frac{0.1 \times 2}{\sqrt{3}},$$

with angular speed

$$\omega = 5 \times 2\pi = 10\pi\ \mathrm{rad\,s^{-1}}.$$

Since the inward acceleration has magnitude $r\omega^2$, the radial component of Newton's second law gives

$$T\sqrt{3} = mr\omega^2.$$

Hence, with $m = 1$ and the values for r, ω found above, we have

$$T\sqrt{3} = \frac{1 \times 0.2 \times 100\pi^2}{\sqrt{3}} \qquad \text{or} \qquad T = \tfrac{20}{3}\pi^2 \simeq 65.8.$$

The tension in each string is therefore 65.8 N.

14. **(i)**

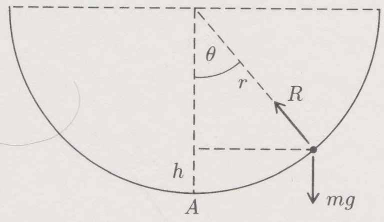


Figure 10

Let θ be the angle between the radius to the particle and the downward vertical, and h the vertical height of the particle above A. The analysis of this situation is identical to that for the pendulum given in Subsection 3.3. From Equation (18) of Section 3, we see that

$$v^2 = v_0^2 - 2gr(1 - \cos\theta), \tag{18}$$

where v_0 is the velocity of the particle at A, and $v = r\dot{\theta}$ is its velocity at angle θ. Initially we have $h = \frac{1}{2}r$, $v = \sqrt{gr}$ and

$$\cos\theta = \frac{r-h}{r} = \tfrac{1}{2},$$

giving

$$gr = v_0^2 - 2gr \times \tfrac{1}{2} \qquad \text{or} \qquad v_0^2 = 2gr.$$

Hence Equation (18) becomes

$$v^2 = 2gr\cos\theta. \tag{19}$$

The speed at the top of the bowl, where $\theta = \pm\frac{1}{2}\pi$, is therefore zero, showing that the particle just reaches this level.

(ii) If R is the magnitude of the reaction from the bowl on the particle, then the component of force acting radially inwards at a given point is $R - mg\cos\theta$. The component of acceleration radially inwards is, using Equation (19),

$$r\dot{\theta}^2 = \frac{v^2}{r} = 2g\cos\theta.$$

Hence the radial component of Newton's second law gives

$$R - mg\cos\theta = 2mg\cos\theta, \qquad \text{or} \qquad R = 3mg\cos\theta.$$

Now with $h = \frac{1}{3}r$, we obtain

$$\cos\theta = \frac{r-h}{r} = \tfrac{2}{3},$$

from which the corresponding reaction force has magnitude

$$R = 2mg.$$

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